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# Three Major Difficulties in the Learning of Demonstrative Geometry

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## PART I
## ANALYSIS OF ERRORS

### CHAPTER I
### PURPOSE AND METHOD

EFFICIENT and successful teaching of demonstrative geometry in the senior high school requires on the part of the teacher much more than a knowledge of the subject matter. The young person who goes into the geometry classroom after leaving college with honors in mathematics is not necessarily a good teacher. Unless he has been forewarned in one way or another, he is likely to resort to the lecture method which his professors have used in college and then find to his surprise that his pupils have learned little. He may have taken courses in which he studied the general laws of learning as applied to pupils of high school age, but even so he will have difficulty in translating his knowledge to fit the specific requirements of the classroom. Part of his training may have been to observe the work of a highly efficient, successful, and artistic teacher whom he may try to imitate. He will find, however, that he has not been keen enough to grasp the meaning and purpose of many of the techniques. Not knowing before hand how a group of pupils will react to a given situation, he fails to see when and how the experienced teacher has avoided pitfalls by introducing many details of development not necessarily needed in the finished product but indispensable to the learning process. Before he can become adept in preparing a course of study or planning his everyday lessons, he needs to know what difficulties pupils will have with the many component tasks which when integrated fulfill the desired aim. A teacher can plan a skillful development only when he has reached a point where he can predict within reasonable limits what the reactions of a group of pupils will be.

A teacher cannot sit in an armchair and by reasoning alone tell how pupils will react to the many situations of the classroom. One who has taught for many years will inevitably know more about pupils' difficulties and the way to remedy, minimize, and obviate them than one who has never taught. But unless he has consciously put his mind to the study of these difficulties and has sufficient background to get meaning from the study, he will have missed one of the best methods of

improving his teaching. The first step in devising methods to overcome pupils' difficulties is to find out what the difficulties are.

The basis for such a study is classroom experience with pupils of various types. For the best results, however, experience alone is insufficient. In order to get the most insight from a study of pupils' learning difficulties one must set up typical situations relevant to the end in view, observe and analyze the responses of individuals. Only when results are recorded systematically are they in shape for careful analysis. And when the learning process requires a rather lengthy period of time as it does in the introduction to demonstrative geometry, the process of setting up situations, observing reactions, and recording the results must continue for the same length of time. A single test or even a series of tests at the end of a learning period will show errors which can be tabulated as to type. A series of tests over a period of time while learning is progressing will give a better insight into the learning process and may lead the investigator to a more basic classification of errors than might otherwise be apparent.

This study is in two parts. Part I is an investigation of pupils' learning difficulties in demonstrative geometry over a period of fifty consecutive teaching days—from the first meeting of the classes through the study of *congruence* and *parallel lines.* During this time, tests were given nearly every day. All the errors were marked and the percentage of pupils making each type of error was recorded. As the study progressed, the writer learned that the number of errors which could be classified under three headings—namely, (1) those due to unfamiliarity with geometric figures, (2) those due to not sensing the meaning of the *if-then relationship,* and (3) those due to a meager understanding of the meaning of proof which far exceeded all other errors. Errors under these headings persisted throughout the study. In the final check of errors by means of a test on "originals" almost all of the errors fell into this classification. For these reasons, the writer believes that these types of error indicate the major learning difficulties in the beginning of demonstrative geometry and has, therefore, restricted the study to a discussion of these three difficulties.

The lack of space given to these three headings in textbooks and the literature on the teaching of geometry suggests that teachers are unaware of the fact that pupils' difficulties are so basic. Suggestions on the teaching of geometry fall short of revealing the fundamental errors. As stated above, the first step toward a method of teaching that will take care of the fundamental weaknesses of the pupils is to become aware of those weaknesses. It is the purpose of Part I of this study, therefore, to take this first step; that is, to identify and analyze the serious learning difficulties that pupils have in connection with (1) geometric figures, (2) the meaning of the if-then relationship, and (3) the meaning of proof.

The study involved all the pupils taking the first course in demonstrative geometry during the fall of 1932 in Classical High School, Springfield, Massachusetts. The number of pupils was 114. All the pupils were members of the 10A class, 10A referring to the second semester of the tenth school year. They were divided into five groups according to their previous record in mathematics, the author teaching two of them in successive periods. The remaining classes were taught by three other experienced teachers. All classes met four times a week for a period of sixty minutes.

Contrary to custom, all the pupils in the five classes studied the same topics and subtopics day by day. So far as was possible, with four teachers involved, the method of teaching was the same in all classes. Bulletins giving detailed information as to topics to be covered each day and methods to be used were given to the teachers, who were cautioned to deal on

any given day with only the topics outlined for that day. Necessarily, the pace of the better classes had to be slackened to meet the requirements of the slower groups. On account of the large number of tests given, the pace was even slower than that ordinarily required for the poorest classes.

The first reading of the test papers was done by the teacher of each class. At this time a cross was placed against each error. The papers were then given to the writer. He analyzed the errors as to type and placed code marks against them to indicate the type of error. The results were then tabulated according to individual pupils and finally according to groups as shown in the following chapters.

The final tabulations give the percentage of pupils making errors in any given case according to four groups: Group A, 30 pupils with I.Q.'s (Terman) from 125 to 146; Group B, 50 pupils with I.Q.'s from 111 to 125; Group C, 34 pupils with I.Q.'s from 90 to 110; and Group T, 114 pupils consisting of Groups A, B, and C together. The median for Group A was 131; for Group B, 117; for Group C, 116; and for Group, T, 116. The mean of I.Q.'s for Group T was 117.9 with a standard deviation of 11.7.

No extended statistical treatment is required to fulfill the purpose of Part I. In every case the percentage of pupils making a particular type of error is given. We are not so much interested in making comparison of the percentages as we are in noting the separate percentages. When a large percentage of the pupils make a certain type of error we see that a learning difficulty and a teaching problem are involved. Comparisons between percentages have been made only where a series of them clearly show a trend or where the difference between two percentages is so large as to be obviously significant. We are not interested in finding out how much more difficult one thing is than another so much as in discovering what is difficult.

Part II is a description of methods devised to help pupils with their difficulties in connection with the three types of errors already mentioned and a study of the effect of these methods on a group of students with the same mean I.Q. and same standard deviation as Group T. The experimentation for Part II was done in the fall of 1938. The method of procedure will be described in full later.

## Chapter II
## COMPLEX FIGURES

Geometric figures are the materials used in the study of demonstrative geometry in the senior high school. They constitute the vehicle which carries the logic of the course. The demonstrations made, the conclusions drawn, all concern geometric figures. Efficient reasoning in geometry presupposes, therefore, a familiarity with such figures. One of the arguments for using geometry to develop methods of reasoning is that the materials with which the pupils reason are simple as compared to those necessary in history, literature, or economics. At the same time we should not be so blinded by this statement that we do not see the need of making sure that pupils are sufficiently familiar with the figures.

It is the tendency in textbooks to define a term, perform a construction, or prove a theorem with the use of as simple a figure as possible and to expect the pupil to apply what he has learned (without help) to more complex figures. We propose to show in this chapter that many pupils do not make the generalization readily.

For the purpose of this study we define a complex figure as one different from the one in which a given term is defined, a given construction is practiced, or a given theorem introduced. A figure may be complicated by adding a line or lines or by

turning it about. Thus, if perpendicular lines are illustrated where defined with a figure showing a horizontal and vertical line, the figure is considered complex if the lines are rotated so that they are oblique to the horizontal. Likewise, it would be complex with respect to *perpendicular* if it involved a triangle with an altitude.

We shall study pupils' reactions to complex figures under three headings: *Constructions*, *Meaning of Terms*, and *Recognition of Theorems*.

### *Constructions*

In order to discover pupils' reactions to constructions in complex figures, they were first shown how to make a construction in a simple situation and were then given practice with it. Then, without further training, they were given construction problems which required the application of what they had learned to a more complex situation. For one fundamental construction problem this procedure was followed by an explanation and a second test given to ascertain the effect of the explanation.

The tests, tabulations of results, and discussions follow.

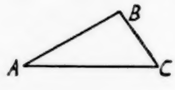
### *Bisecting Lines*

Test 1[1] was given on the third day, after the pupils had learned how to bisect horizontal straight lines and had practiced to the point of mastery. The purpose of the test was to discover whether the ability to bisect a horizontal straight line could be used in bisecting the sides of a triangle.

TEST 1

1. Bisect $AB$.


2. Bisect the sides of triangle $ABC$.




[1] Tests are numbered and recorded not in the order given, but in an order that furthers an understanding of the discussion. Only those tests which have a direct bearing on the topics selected are recorded.

TABLE 1

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 1

| Exercise | Percentage in Group[2] | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 4 | 15 | 6 |

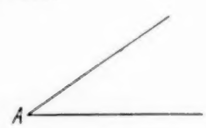
There were no errors on Ex. 1 showing that, for the time being at least, all the pupils could bisect a horizontal straight line correctly. The transfer from Ex.1 to Ex. 2 was 100% in case of the upper I.Q. group (Group A), but dropped somewhat in the lower groups. In the middle group (Group B) 4% of the pupils were in error and in the lowest group (Group C) 15%. We may consider the transfer as high, since 94% of the entire group were correct on Ex. 2.

### *Bisecting Angles*

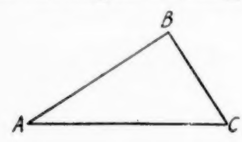
Test 2 was given on the fifth day. Pupils had practiced bisecting angles with the vertex at the left and the opening at the right, but had not bisected any other angles. The purpose of the test was similar to that of Test 1; namely, to discover whether the ability to bisect an angle given in a special position could be used in bisecting the angles of a triangle.

TEST 2

1. Bisect $\angle A$.




2. Bisect the angles of triangle $ABC$.




[2] Group A comprised 30 pupils with I.Q.'s from 126–146; Group B, 50 with I.Q.'s from 111–125; and Group C, 34 with I.Q.'s from 90 to 110. Group T was a combination of Groups A, B, and C (see page 101).

TABLE 2

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 2

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 10 | 12 | 23 | 15 |

The following types of error were made on Ex. 2 of Test 2.

1. Bisection of $\angle A$ correct, bisection of other angles confused.

2. Bisection of all angles confused.

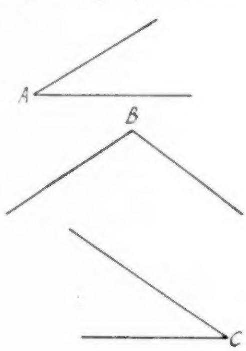
3. Bisection of two angles correct. Bisection of third angle confused.

The third type of error accounted for two-thirds of all errors. Confusion showed in not drawing the first arc across the sides of the angle to be bisected. The arc was sometimes drawn across one side of one angle and one side of another angle. The fact that the line did not appear to bisect the angle seemed to cause no anxiety.

As soon as Test 2 was finished, Test 2a was given to ascertain whether the difficulty in Ex. 2 of Test 2 was caused by the different positions of the angles.

TEST 2a

Bisect angles $A$, $B$, and $C$.




Since all the pupils were correct on the three exercises, it may be concluded that the difficulty was not the position of the angles. The cause of difficulty was obviously the confusion caused by the many arcs and lines. Pupils had not generalized the method of bisecting an angle sufficiently to make sure that their first arc in each case intersected the two sides of the angle being bisected. This conclusion is borne out by the fact that after pupils had discussed the method of bisecting an angle in general; that is, so that it applied to no particular figure, the errors were almost entirely eliminated.

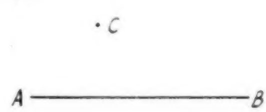
*Constructing Perpendiculars*

The next test (on the construction of perpendiculars) was given on the eighth day after pupils had practiced with Exs. 1–4 of the Test. In this test we find more varied complications of figures than in Tests 1 or 2 and consequently greater variations in results. It was given after Test 6 (see page 107) but before Test 6 was discussed.

TEST 3

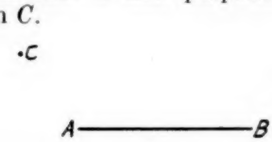
1. Construct a line perpendicular to $AB$ at $C$.


2. Construct a line perpendicular to $AB$ from $C$.

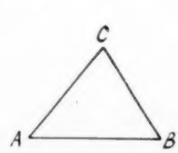



3. Construct a line perpendicular to $AB$ at $A$.

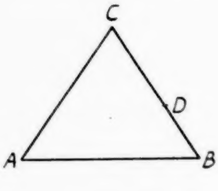

4. Construct a line perpendicular to $AB$ from $C$.



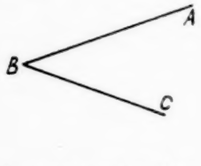

5. Construct a line from $B$ perpendicular to $AC$.



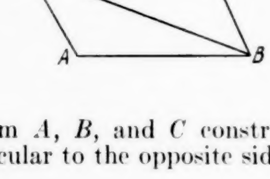
6. Construct a line from $D$ perpendicular to $AC$.



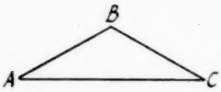

7. At $B$ construct a line perpendicular to $AB$ and another perpendicular to $BC$.



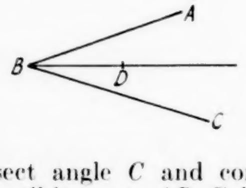

8. From $C$ construct a line perpendicular to $DB$.



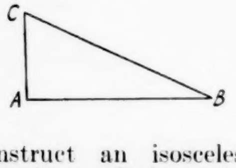

9. From $A$, $B$, and $C$ construct lines perpendicular to the opposite sides of the triangle.




10. From $D$ construct a line perpendicular to $AB$.




11. Bisect angle $C$ and continue the bisector until it meets $AB$. Call the point of intersection $D$. From $D$ construct a line perpendicular to $CB$.




12. Construct an isosceles triangle. From any point on the base of the triangle construct lines perpendicular to the equal sides.

TABLE 3

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 3

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 2 | 0 | 1 |
| 4 | 3 | 0 | 0 | 1 |
| 5 | 10 | 12 | 15 | 12 |
| 6 | 13 | 10 | 12 | 11 |
| 7 | 10 | 12 | 24 | 15 |
| 8 | 3 | 8 | 12 | 8 |
| 9 | 13 | 26 | 26 | 23 |
| 10 | 7 | 10 | 12 | 10 |
| 11 | 17 | 20 | 26 | 21 |
| 12 | 20 | 36 | 59 | 39 |

The following types of error were made.

1. Perpendicular drawn to the wrong line.
2. Perpendicular at or from the wrong point.
3. Wrong method of construction resulting in a line obviously not perpendicular.
4. Incomplete.
5. Angles bisected in Ex. 9.
6. Perpendicular bisectors of the sides constructed in Ex. 9.

There were no errors in Exs. 1 and 2 and the number of errors in Exs. 3 and 4 was negligible. These were the practice exercises. In these only one point and one line were involved. In the others, although the point and the line were explicitly stated, many pupils could not dissociate them from the rest of the figure and were confused. The reader should note how a very little change in the situation (the figures) causes a change in the results. For example, Ex. 7 requires the pupil do twice what he has already done correctly in Ex. 3, the main difference being the fact that when he is constructing the perpendicular to $AB$ he must ignore $BC$, and when constructing the perpendicular to $BC$ he must ignore $AB$. Nevertheless 15% of the pupils could not do Ex. 7 correctly. In Ex. 9, where there were three perpendiculars to construct and two of them required the extension of a line, 23% of the

pupils were in error. (A very few of these were pupils who did not follow directions as stated in the preceding list.) This was in spite of the fact that they had demonstrated their ability to take care of the elements of this exercise by doing Exs. 2 and 4 correctly.

In the twelfth exercise we see what is likely to happen when the problem is further complicated by being stated in words without an accompanying figure. Thirty-nine per cent (59% in Group C) of the pupils could not do this exercise.

In subsequent class work it was found that the errors due to the complexity of the figures could be almost entirely eliminated by generalizing the method of construction. By analysis of the problem pupils were led to see that in every case the point of the compasses should be placed on the given point and the first arc drawn so that it would intersect the given line twice. If the line was not sufficiently long so that the arc would intersect it twice, it had to be extended. (See Part II of this study.)

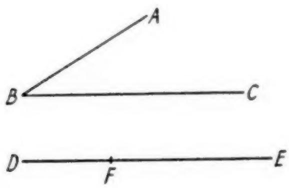
### *Constructing an Angle Equal to a Given Angle*

The next two tests concern the construction of an angle equal to a given angle. In these we meet a new complication. Pupils were confronted for the first time with the necessity of thinking of the order of drawing lines in constructing a complex figure. The reader will see that this complication caused more errors than were found in any preceding test.

Test 4 was given on the tenth day. Pupils were familiar with angles and with the reading of angles and had practiced making an angle equal to a given angle with all lines in the position as shown in Ex. 1 until all but 4% (see Table 4) had mastered the construction.

### Test 4

1. Construct an angle equal to angle $ABC$ using $F$ as vertex and $FE$ as one side.




2. Construct a figure like the one below so that angle 1 will be equal to angle 2. In constructing the figure what line did you draw first? What line did you draw next?

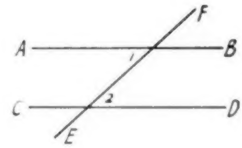



TABLE 4
Percentage of Pupils Making Errors on Each Exercise of Test 4

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 6 | 3 | 4 |
| 2 | 63 | 80 | 71 | 73 |

The actual construction required in Ex. 2 was the same as that in Ex. 1, but the pupils were under the necessity of drawing the lines in the correct order. An analysis of the results shows that 36% drew the lines in an impossible order for completion of the construction, 29% were entirely confused and 8% had wrong constructions for the angles.

On the following day, the construction of an angle equal to a given angle was discussed from the general point of view and Ex. 2 of Test 4 was discussed as follows.

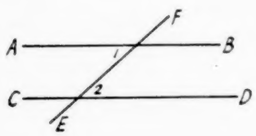
How many lines are there? What angles are we required to make equal? Suppose I draw $AB$ and $CD$ first and then draw $EF$. The angles 1 and 2 are already made and I have not constructed them equal. (This was illustrated at the board.) This shows that the order of drawing the lines is important. In order to construct an angle equal to another, you must first draw one of them. In this case draw either

$\angle 1$ or $\angle 2$ before you try to construct the other.

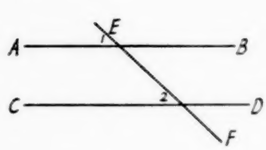
The discussion was left here. Pupils were not shown how to construct the figure. Test 5 was then given to discover the reactions of pupils after the kind of discussion reported above.

### Test 5

1. Construct a figure like the one below right, so that $\angle 1$ will equal $\angle 2$. What line did you draw first? What line did you draw next?




2. Construct a figure like the one below right, so that $\angle 1$ will equal $\angle 2$. What line did you draw first? What line did you draw next?




3. Given triangle $ABC$. Construct another triangle $DEF$ so that $DE$ will equal $AB$, $EF$ will equal $BC$, and $\angle E$ will equal $\angle B$.

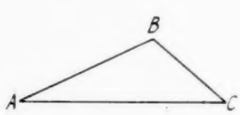



TABLE 5

Percentage of Pupils Making Errors on Each Exercise of Test 5

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 23 | 34 | 33 | 31 |
| 2 | 37 | 48 | 56 | 47 |
| 3 | 73 | 80 | 85 | 80 |

The effect of the very brief discussion of Ex. 1 is seen in the comparison of the number of pupils who did this exercise incorrectly in Test 4 and in Test 5. In Test 4, 73% of the pupils were wrong. In Test 5 the corresponding per cent was 31. Only 3% drew the lines in the wrong order as compared to 36% in Test 4. There was little evidence of complete confusion as in the preceding test. The most frequent error was that of making the wrong angles equal (10%).

A very slight change in the problem from Ex. 1 to Ex. 2 caused the number of pupils in error to jump from 31% to 47%. In Ex. 2, 4% drew the lines in the wrong order and 15% made the wrong angles equal. And a seemingly very simple exercise (Ex. 3) so far as construction is concerned caused nearly double the number of errors (80%). In this exercise 29% were entirely confused as to method of procedure (the order of drawing the lines was again the probable cause), and 40% were wrong because they made either two angles and the included side or three sides of one equal to the corresponding parts of the other instead of two sides and the included angle.

The tendency (shown in Ex. 3) of many pupils to construct a figure, not according to directions, but so that it looks right in the end will be discussed in connection with the if-then relationship in the next chapter.

From the results of the foregoing tests we see that mere ability to perform a construction in a particular situation is not necessarily sufficient to assure ability to make the same construction in a slightly different situation. A small change in the situation causes difficulty. Pupils are confused by additional lines or the moving about of a figure. They do not dissociate the essential and particular parts to which they should give attention from the parts of a figure they should disregard. And when it comes to a figure which requires analysis to know what parts are to be drawn first, what second, and so on, the difficulties (if instruction is not given) are overwhelming.

### *Meaning of Terms*

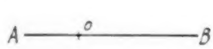
In order to discover pupils' reactions to the meaning of terms in complex figures the procedure employed in connection with constructions was used. Pupils were trained as to the meaning of certain terms to the point of mastery (or nearly so) with a simple figure and then tested on the meaning of these same terms using complex figures. The results of four tests are given here: one on *perpendiculars;* two on the terms *two sides and the included angle* and *two angles and the included side;* and one on *alternate interior angles* in connection with *parallel lines.*

### *Perpendiculars*

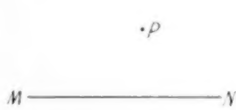
#### TEST 6

Use rulers only.

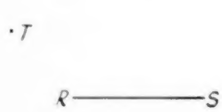
1. Draw a line perpendicular to $AB$ at $O$.



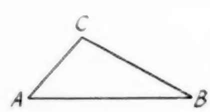

2. Draw a line perpendicular to $MN$ from $P$.




3. Draw a line perpendicular to $RS$ from $T$.

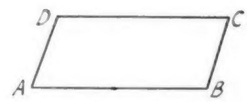



4. Draw a line from $C$ perpendicular to $AB$.

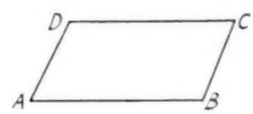



5. Draw lines perpendicular to $AB$ at $A$ and at $B$.

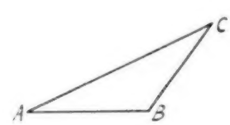

6. Draw a line from $D$ perpendicular to $AB$.



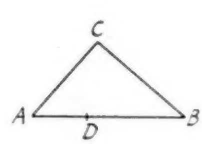

7. Draw a line from $C$ perpendicular to $AB$.



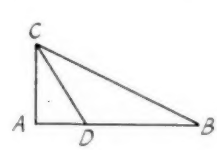

8. From $B$ draw a line perpendicular to $AC$.



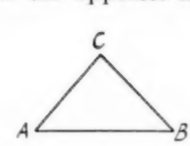

9. From $D$ draw a line perpendicular to $BC$.



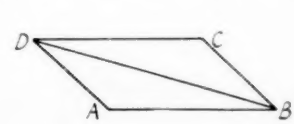

10. From $D$ draw a line perpendicular to $BC$.



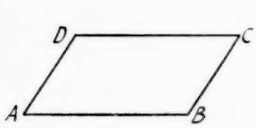

11. From $A$, $B$, and $C$ draw lines perpendicular to the opposite sides.



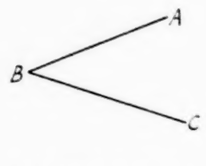

12. From $C$ draw a line perpendicular to $DB$.



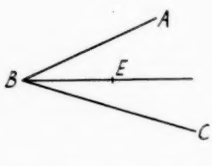
13. At $B$ draw a line perpendicular to $BC$.



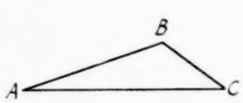

14. At $B$ draw a line perpendicular to $AB$ and another perpendicular to $BC$.



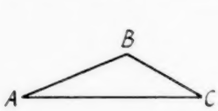

15. From $E$ draw a line perpendicular to $AB$.



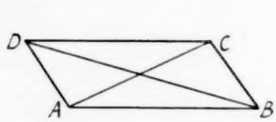

16. From $C$ draw a line perpendicular to $AB$.




17. From $A$, $B$, and $C$ draw lines perpendicular to the opposite sides.




18. From the ends of $AC$. draw lines perpendicular to $DB$.




This test was given on the sixth day (see the first footnote on page 102) after the meaning of *perpendicular* had been discussed and illustrated as in Exs. 1–3. Pupils had practiced drawing perpendiculars to horizontal lines as in these same three exercises. The purpose of the test was to discover whether the ability to draw perpendiculars to horizontal lines as in the practice exercises could be used in drawing perpendiculars in complex figures.

TABLE 6

Percentage of Pupils Making Errors on Each Exercise of Test 6

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 3 | 0 | 0 | 1 |
| 4 | 3 | 0 | 9 | 4 |
| 5 | 3 | 2 | 6 | 4 |
| 6 | 3 | 10 | 6 | 7 |
| 7 | 7 | 2 | 3 | 4 |
| 8 | 10 | 12 | 15 | 12 |
| 9 | 7 | 26 | 21 | 19 |
| 10 | 3 | 21 | 18 | 16 |
| 11 | 7 | 14 | 12 | 11 |
| 12 | 7 | 26 | 24 | 20 |
| 13 | 37 | 48 | 47 | 45 |
| 14 | 20 | 38 | 36 | 32 |
| 15 | 10 | 36 | 29 | 27 |
| 16 | 23 | 40 | 65 | 43 |
| 17 | 27 | 60 | 65 | 52 |
| 18 | 37 | 54 | 41 | 46 |

The practice exercises 1–3 required in each case a perpendicular to a horizontal line, and there was no other line to cause confusion. Exs. 4–7 also required perpendiculars to horizontal lines, but there were slight complications. The increase in the number of errors (to 4 and 7%) was not great but was consistent. In Exs. 8–12 pupils were asked to draw perpendiculars to lines that were not horizontal and immediately the number of errors increased considerably. The percentages of pupils in error on these exercises ranged from 11 to 20. The most frequent error was to draw a perpendicular to the wrong line or to draw it so that it was obviously not perpendicular. In the next two exercises, 13 and 14, we find a new complication. The perpendiculars were to be drawn at a point on a line where a second line met it. The percentage in error jumped to 45 in the thirteenth exercise and 32 in the fourteenth. Most of the errors on Ex. 15 were due to the fact that pupils drew a line

perpendicular to the horizontal line $BE$ instead of to $AB$. Exs. 16 and 17 required the extension of a line before the perpendicular could be drawn just as did Ex. 3. However, these two exercises required the drawing of perpendiculars to lines that were not horizontal and the figures contained lines which had to be ignored while drawing a particular perpendicular. While the percentage of pupils in error on Ex. 3 was only 1, the percentage on Ex. 16 was 43 and on Ex. 17 was 52. Ex. 18 was included for comparison with Ex. 12. The figure for the latter had two diagonals instead of one, and the language of the directions for it required more careful reading. Otherwise the two exercises were nearly the same. Yet the percentage in error on the eighteenth exercise was 46 as compared to 20 on the twelfth.

*Choosing s.a.s. and a.s.a.*

The next two tests, 7 and 8 as recorded here, were given on the forty-fifth day. Pupils had been proving triangles congruent for several days by means of the *two sides and the included angle* (s.a.s.) and *two angles and the included side* (a.s.a.) relations, but had had no specific practice in choosing these combinations, and had had no work at all with overlapping triangles. The purpose of Test 7 was to discover to what extent pupils would make errors in choosing s.a.s. and a.s.a. in figures of varying degrees of complexity, particularly in figures containing overlapping triangles.

Before the tests were given, a triangle $ABC$ was drawn on the board, and the following exercises discussed.

Fill in the blanks so that the results will be—

1. s.a.s. of $\triangle ABC$. $AB, \cdots, BC$.
2. a.s.a. of $\triangle ABC$. $\angle A, \cdots, \angle C$.
3. a.s.a. of $\triangle ABC$. $\cdots, BC, \cdots$
4. s.a.s. of $\triangle ABC$. $\cdots, \angle C, \cdots$

Teachers were asked to continue the practice until they were reasonably sure that all pupils understood what they were expected to do.

TEST 7

Fill in the blanks so that the results will be—

1. s.a.s. of $\triangle ABD$ (Fig. 1). $BD, \angle 3, \cdots$
2. s.a.s. of $\triangle DBC$ (Fig. 1). $BC, \angle 2, \cdots$
3. a.s.a. of $\triangle ABD$ (Fig. 1). $\angle A, \cdots, \angle 1$.
4. a.s.a. of $\triangle BDC$ (Fig. 1). $\angle 2, \cdots, \angle 4$.

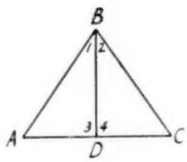


FIG. 1.

5. s.a.s. of $\triangle ABC$ (Fig. 2). $BC, \cdots, AC$.
6. a.s.a. of $\triangle ACD$ (Fig. 2). $\angle 2, \cdots, \angle 4$.
7. s.a.s. of $\triangle ACD$ (Fig. 2). $\cdots, \angle 4, \cdots$
8. a.s.a. of $\triangle ACD$ (Fig. 2). $\cdots, AC, \cdots$

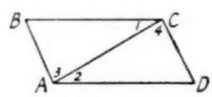


FIG. 2.

9. s.a.s. of $\triangle COD$ (Fig. 3). $CD, \cdots, DO$.
10. a.s.a. of $\triangle AOB$ (Fig. 3). $\angle 3, \cdots, \angle 2$.
11. s.a.s. of $\triangle AOB$ (Fig. 3). $\cdots, \angle 2, \cdots$
12. a.s.a. of $\triangle COD$ (Fig. 3). $\angle 1, DC, \cdots$

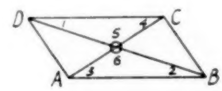


FIG. 3.

13. s.a.s. of $\triangle ACE$ (Fig. 4). $AC, \cdots, CE$.
14. a.s.a. of $\triangle BCD$ (Fig. 4). $\cdots, CD, \cdots$

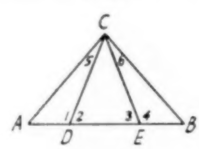


FIG. 4.

15. s.a.s. of $\triangle CAE$ (Fig. 5). $AC, \cdots, AE$.
16. s.a.s. of $\triangle CAD$ (Fig. 5). $AC, \angle A, \cdots$
17. a.s.a. of $\triangle ACD$ (Fig. 5). $\cdots, AC, \cdots$
18. a.s.a. of $\triangle ACE$ (Fig. 5). $\cdots, AC, \cdots$
19. s.a.s. of $\triangle AEB$ (Fig. 5). $\cdots, \angle B, \cdots$
20. s.a.s. of $\triangle CDB$ (Fig. 5). $\cdots, \angle B, \cdots$

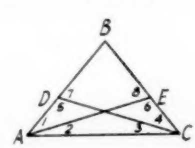


FIG. 5.

TABLE 7

Percentage of Pupils Making Errors on Each Exercise of Test 7

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 2 | 3 | 3 |
| 2 | 0 | 0 | 3 | 1 |
| 3 | 0 | 4 | 12 | 5 |
| 4 | 0 | 4 | 6 | 4 |
| 5 | 10 | 10 | 16 | 11 |
| 6 | 0 | 6 | 6 | 4 |
| 7 | 0 | 4 | 9 | 4 |
| 8 | 3 | 4 | 12 | 6 |
| 9 | 3 | 6 | 12 | 7 |
| 10 | 0 | 0 | 6 | 2 |
| 11 | 3 | 2 | 6 | 4 |
| 12 | 13 | 16 | 12 | 14 |
| 13 | 3 | 28 | 41 | 25 |
| 14 | 3 | 34 | 47 | 30 |
| 15 | 10 | 10 | 21 | 13 |
| 16 | 3 | 4 | 24 | 10 |
| 17 | 23 | 32 | 68 | 40 |
| 18 | 17 | 26 | 56 | 32 |
| 19 | 3 | 8 | 18 | 10 |
| 20 | 0 | 4 | 15 | 6 |

Exs. 13–20 involved overlapping triangles. The percentages of pupils in error on these exercises ranged from 6 to 40, half of which were 25 or higher as compared with a range from 1 to 14 of which all but two were 7 or below on the first twelve exercises. Throughout the test the exercises requiring a choice of sides caused fewer errors than those requiring the choice of angles.

### Test 8

1. Trace the three sides of triangle *BCD* with a colored pencil. Fill in the blanks so that the result will be a.s.a. of triangle *BCD*. (Fig. 1)

· · ·, *CD*, · · ·

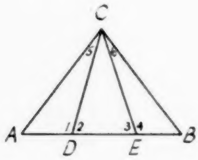



Fig. 1.

2. Trace the three sides of triangle *ACE* with a colored pencil. Fill in the blank so that the result will be s.a.s. of triangle *ACE*. (Fig. 2)

*AC*, · · · , *CE*

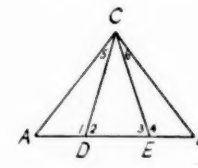



Fig. 2.

3. Trace the three sides of triangle *ACE* with a colored pencil. Fill in the blanks so that the result will be a.s.a. of triangle *ACE*. (Fig. 3)

· · · , *AC*, · · ·

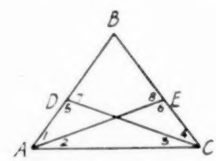



Fig. 3.

4. Trace the three sides of triangle *AEB* with a colored pencil. Fill in the blanks so that the result will be s.a.s. of triangle *AEB*. (Fig. 4).

· · · , ∠*B*, · · ·

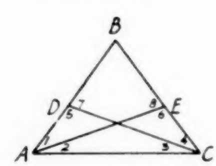



Fig. 4.

TABLE 8

Percentage of Pupils Making Errors on Each Exercise of Test 8

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 18 | 32 | 18 |
| 2 | 3 | 21 | 29 | 19 |
| 3 | 7 | 14 | 29 | 17 |
| 4 | 0 | 4 | 3 | 3 |

Except for the tracings with colored pencils, Exs. 1, 2, 3, and 4 of Test 8 were identical respectively with Exs. 14, 13, 18, and 19 of Test 7. Comparison of the results of these two sets of exercises shows that the use of colored pencils was effective.

Pupils could isolate a particular triangle when it was brought out by being traced with a colored pencil more easily than when not. The percentages of pupils in error on these exercises in Test 7 (with no tracing) were 30, 25, 32, and 10; on Test 8 (with tracing) the corresponding percentages were consistently lower, being 18, 19, 17, and 3. Even so, the percentages in error continued to be higher than in those exercises of Test 7 which did not involve overlapping triangles.

Again as in Test 7, it was found easier to choose a side (see Ex. 4) than to chose angles.

*Alternate Interior Angles*

The next test (Test 9) was given on the forty-ninth day. Pupils knew the definition of alternate interior angles, but had had experience with them only in case of a figure involving two parallel lines and a transversal (see figure for Ex. 1).

TEST 9

Read carefully. If the lines are parallel as indicated in each of the following exercises, what *alternate interior angles* are equal? (There may be more than one pair.) Supply numbers in the angles if you need them.

1. (Fig. 1). If $AB$ is parallel to $CD$,

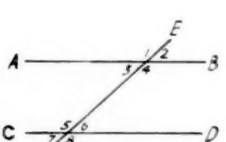



FIG. 1.

2. (Fig. 2). If $AB$ is parallel to $CD$,
3. (Fig. 2). If $AD$ is parallel to $BC$,

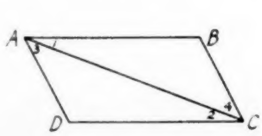



FIG. 2.

4. (Fig. 3). If $DE$ is parallel to $AC$,

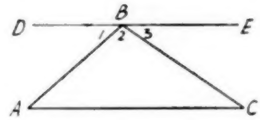



FIG. 3.

5. (Fig. 4). If $AB$ is parallel to $CD$,

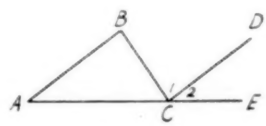



FIG. 4.

6. (Fig. 5). If $AD$ is parallel to $BC$,
7. (Fig. 5). If $AB$ is parallel to $CD$,

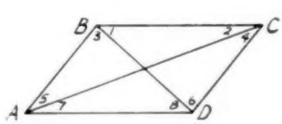



FIG. 5.

8. (Fig. 6). If $AB$ is parallel to $CF$,

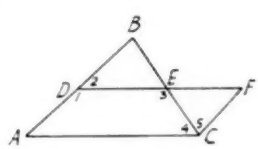



FIG. 6.

9. (Fig. 7). If $AE$ is parallel to $BD$,

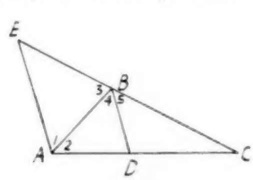



FIG. 7.

10. (Fig. 8). If $BC$ is parallel to $AF$,

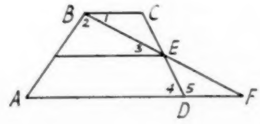



FIG. 8.

TABLE 9

Percentage of Pupils Making Errors on Each Exercise of Test 9

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 6 | 6 | 5 |
| 2 | 10 | 24 | 38 | 25 |
| 3 | 7 | 24 | 38 | 24 |
| 4 | 14 | 38 | 44 | 33 |
| 5 | 17 | 24 | 35 | 25 |
| 6 | 7 | 38 | 47 | 32 |
| 7 | 7 | 38 | 53 | 34 |
| 8 | 63 | 62 | 82 | 68 |
| 9 | 23 | 48 | 53 | 43 |
| 10 | 63 | 78 | 97 | 80 |

Typical errors on each exercise with the percentage of pupils making them, beginning with Ex. 2, are listed below.

Ex. 2. $\angle 1=\angle 2$, $\angle 3=\angle 4$, 15%.

Ex. 3. $\angle 1=\angle 2$, $\angle 3=\angle 4$, 14%.

Ex. 4. $\angle 1=\angle A$ only, 7%; $\angle 3=\angle C$ only 5%; $\angle 1=\angle C$, $\angle 3=\angle A$, 5%; $\angle 1=\angle 3$, 4%.

Ex. 5. $\angle 1=\angle A$, 4%; $\angle A=\angle 2$, 4%; none, 3%; $\angle 1=\angle 2$, 3%; $\angle B=\angle C$, 3%.

Ex. 6. Four pairs of angles equal, 14%.

Ex. 7. Four pairs of angles equal, 10%.

Ex. 8. $\angle B=\angle 5$ only, 48%; $\angle 2=\angle F$ only, 5%.

Ex. 9. $\angle 3=\angle 2$, $\angle 1=\angle 4$, 25%; none, 5%.

Ex. 10. $\angle 1=\angle F$ only, 55%.

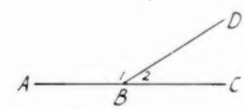
If we consider as correct those exercises where one pair of angles was given correctly and the others not mentioned, the percentages of pupils in error would read as follows for Group T: Ex. 4 (21%), Ex. 6 (26%), Ex. 7 (29%), Ex. 8 (15%), and Ex. 10 (25%). The errors of omission on Exs. 8 and 10 far outnumbered the errors of commission.

The figure of Ex. 1 was that with which the definition of alternate interior angles was given. Only 5% of the pupils made errors on this exercise. There was a decided increase in the number of pupils making errors on all the other exercises, as will be seen by a mere glance at the results recorded in Table 9.

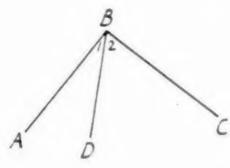
*Recognition of the Application of Theorems in Complex Figures*

We have already shown that even though a pupil may be able to perform a given construction in a simple figure he may not be able to perform the same construction in a complex figure and that terms which have meaning to him in a simple figure may not be clear to him in a complex figure. In this section of the chapter we propose to show that a similar conclusion may be drawn in regard to a pupil's recognition of the application of theorems in complex figures.

The following test (Test 10) was given on the twenty-sixth day. Pupils were familiar with the proposition: When one straight line meets another so as to form adjacent angles these angles are supplementary, in connection with a figure as in Ex. 1, but with no other figure. Before the test was given the following two figures were placed on the board, discussed from the point of view of the theorem, and left there while the pupils took the test.




$\angle 1$ is supplementary to $\angle 2$.




No.

The discussion of the two figures was as follows: In the first figure we have one straight line meeting another so as to form adjacent angles 1 and 2, so we write under it, "$\angle 1$ is supplementary to $\angle 2$." In the second figure we have adjacent angles, but they are not formed by one straight line meeting another. Hence under this figure, we write, "No."

## Test 10

*If one straight line meets another so as to form adjacent angles, these angles are supplementary.*

In some of the figures on this paper, it is possible to apply the above theorem *one or more times*. In some of the figures it is not possible to apply the theorem. When it is possible, tell what angles are supplementary. When it is not possible, write the word "No."

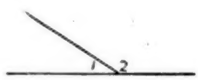


Fig. 1.

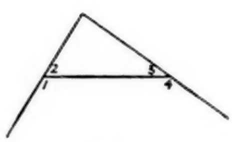


Fig. 2.

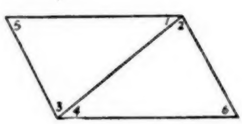


Fig. 3.

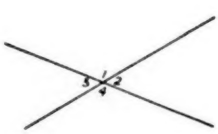


Fig. 4.

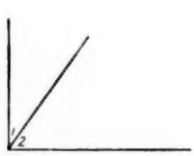


Fig. 5.

Fig. 6.

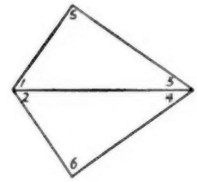


Fig. 7.

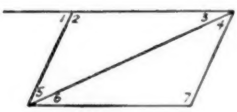


Fig. 8.

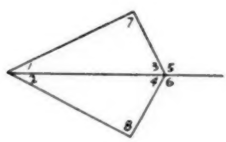


Fig. 9.

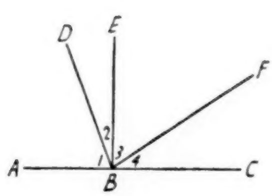



Fig. 10.

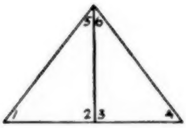


Fig. 11.

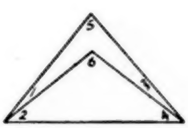


Fig. 12.

Overlapping angles caused serious difficulty in Ex. 10. In this exercise 82% of Group C and 67% of Group T were in error. The most frequent error was: $\angle 1$ is supplementary to $\angle 2$, $\angle 2$ to $\angle 3$, and $\angle 3$ to $\angle 4$ (30%). Exs. 2, 6, 9, and 11 caused appreciable difficulty, especially with

TABLE 10

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 10

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 7 | 4 | 15 | 8 |
| 3 | 0 | 4 | 0 | 2 |
| 4 | 0 | 0 | 0 | 0 |
| 5 | 0 | 2 | 0 | 1 |
| 6 | 7 | 10 | 12 | 10 |
| 7 | 0 | 0 | 3 | 1 |
| 8 | 0 | 4 | 3 | 3 |
| 9 | 3 | 4 | 9 | 5 |
| 10 | 53 | 64 | 82 | 67 |
| 11 | 0 | 4 | 9 | 4 |
| 12 | 0 | 2 | 6 | 3 |

Group $C$. The most frequent errors were: Ex. 2, "No"; Ex. 6, "No"; Ex. 9, $\angle 3$ is supplementary to $\angle 4$ and $\angle 5$ to $\angle 6$. In Ex. 11 there were miscellaneous errors such as No, $\angle 5$ is supplementary to $\angle 6$, and $\angle 1$ is supplementary to $\angle 4$. Except for Ex. 10, however, the transfer was high.

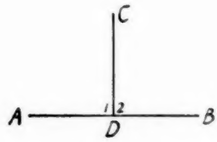
The following test (Test 11) was given on the thirty-first day. The purpose was to discover to what extent pupils would recognize applications of the given three statements in the given figures. Exs. 1,2, and 3 were practice exercises.

### TEST 11

Read the three statements written below. Then look at each figure (and the given statement, if there is one), and decide whether any one of the three statements applies to it. If one of the statements does apply to a given figure, tell what angles are therefore equal and tell which statement applies. If the statements do not apply, write the word "No." Assume that all the lines are straight.

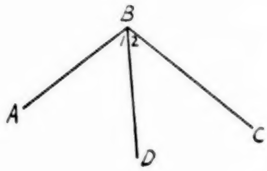
1. *A bisector divides an angle into two equal angles.*
2. *The adjacent angles at the foot of a perpendicular are equal.*
3. *If two straight lines intersect, the opposite angles are equal.*

Examples. In the figure below you are told that $CD$ is perpendicular to $AB$. Statement 2 applies and $\angle 1 = \angle 2$. You should therefore write under the figure, "$\angle 1 = \angle 2$ (statement 2)."




$CD \perp AB$.

In the figure below none of the three statements applies. You should therefore write under the figure the word "No."




Proceed similarly with the figures below.

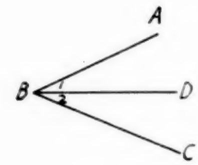



FIG. 1.
$CD$ bisects $\angle B$.

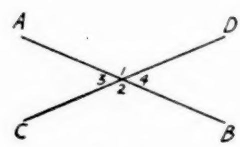



FIG. 2.

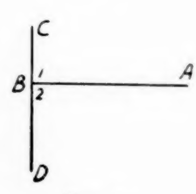



FIG. 3.
$AB \perp CD$.

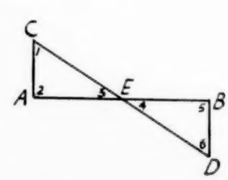



FIG. 4.

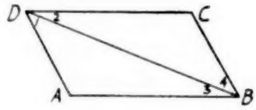



Fig. 5.

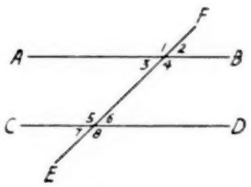



Fig. 6.

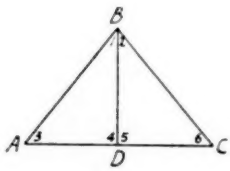



Fig. 7.
$BD \perp AC$.

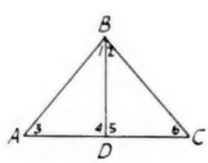



Fig. 8.
$BD$ bisects $\angle B$.

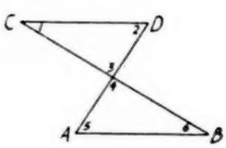



Fig. 9.

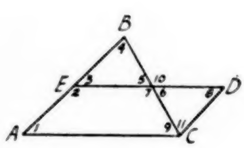



Fig. 10.

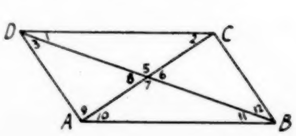



Fig. 11.

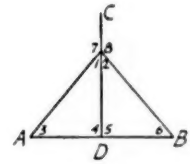



Fig. 12.
$CD$ bisects $AB$.

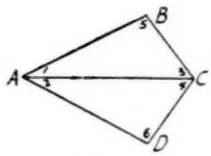



Fig. 13.
$AB = AD$.

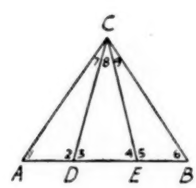



Fig. 14.

TABLE 11

Percentage of Pupils Making Errors on Each Exercise of Test 11

| Exercise | Percentage in Group | | | |
| --- | --- | --- | --- | --- |
| | A | B | C | T |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 3 | 22 | 18 | 16 |
| 5 | 10 | 32 | 24 | 24 |
| 6 | 7 | 12 | 24 | 14 |
| 7 | 7 | 8 | 15 | 10 |
| 8 | 23 | 22 | 38 | 27 |
| 9 | 3 | 18 | 6 | 11 |
| 10 | 3 | 22 | 21 | 17 |
| 11 | 10 | 22 | 24 | 19 |
| 12 | 53 | 82 | 79 | 73 |
| 13 | 23 | 50 | 41 | 40 |
| 14 | 13 | 14 | 6 | 11 |

Typical errors were as follows:

Ex. 4. Angles chosen equal because they appeared equal in the figure, statement 1 or 2 given as reason, 8%. "No," 6%. None of the 16% in error recognized the opposite angles.

Ex. 5. Angle bisectors assumed by 19%.

Ex. 6. "No," 7%. $\angle 1 = \angle 2$, etc., for various reasons, 7%.

Ex. 7. $\angle 1 = \angle 2$, $\angle 4 = \angle 5$ (statement 2). 7%.

Ex. 8. $\angle 4 = \angle 5$, $\angle 1 = \angle 2$ (statement 2), 10%. $\angle 4 = \angle 5$ (statement 1 or 2), 10%. "No," 3%.

Ex. 9. "No," 7%.

Ex. 10. "No," 10%. Omitted, 4%.

Ex. 11. Assumed angle bisectors, 8%. "No," 6%.

Ex. 12. $\angle 4 = \angle 5$ (statement 2), 26%. $\angle 1 = \angle 2$ (statement 1), 14%. $\angle 4 = \angle 5$ (statement 1), 13%. Several pairs equal (statement 1), 14%.

Ex. 13. Assumed angle bisectors, 32%. Omitted, 6%.

Ex. 14. Omitted, 10%.

Most of the errors on this test were due to a cause not under discussion in this chapter. They were due to the pupils' assuming equality of angles from the appearance of the figure and their lack of understanding that they might correctly draw conclusions only from the data. We shall therefore return to this test in Chapter IV. For our present purposes we should see that five figures (Exs. 4, 6, 9, 10, and 11) in addition to the second figure called for the recognition of opposite angles made by two intersecting lines and that the number of pupils in error on these exercises ranged from 11 to 19 per cent. This in spite of the fact that there were no errors on Ex. 1.

Test 12 was given on the forty-third day after pupils had been using *two sides and the included angle* and *two angles and the included side* in proving triangles congruent for several days. They had not, however, been cautioned against combinations that were neither of these. The purpose of the test was to discover to what extent pupils would realize that certain given combinations of sides and angles in triangles were neither *two sides and the included angle* nor *two angles and the included side*.

### Test 12

Read the hypothesis of each exercise, then basing your answers only on the two theorems following, answer the questions. (Use the symbols s.a.s. = s.a.s. and a.s.a. = a.s.a.)

If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.

If two angles and the included side of one triangle are equal respectively to two angles and the included side of another, the triangles are congruent.

1. (Use Fig. 1.) Hyp. $AB = BC$, $BD = BD$. Are the triangles congruent? If so, why?
2. (Use Fig. 1.) Hyp. $AB = BC$, $BD = BD$, $\angle A = \angle C$. Are the triangles congruent? If so, why?

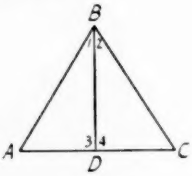



Fig. 1.

3. (Use Fig. 1.) Hyp. $AD = DC$, $BD = BD$, $\angle 3 = \angle 4$. Are the triangles congruent? If so, why?
4. (Use Fig. 1.) Hyp. $\angle 1 = \angle 2$, $\angle A = \angle C$, $BD = BD$. Are the triangles congruent? If so, why?
5. (Use Fig. 2.) Hyp. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $BD = BD$. Are the triangles congruent? If so, why?
6. (Use Fig. 2.) Hyp. $AD = BC$, $BD = BD$, $\angle 3 = \angle 4$. Are the triangles congruent? If so, why?

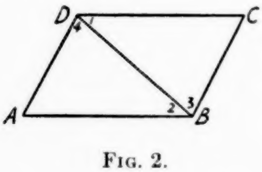



Fig. 2.

7. (Use Fig. 2.) Hyp. $BD = BD$, $\angle 1 = \angle 2$, $AD = BC$. Are the triangles congruent? If so, why?
8. (Use Fig. 2.) Hyp. $\angle 1 = \angle 4$, $\angle 3 = \angle 2$, $BD = BD$. Are the triangles congruent? If so, why?
9. (Use Fig. 3.) Hyp. $\angle 1 = \angle 6$, $\angle 4 = \angle 2$, $AB = DC$. Are triangles $AOB$ and $DOC$ congruent? If so, why?

10. (Use Fig. 3.) Hyp. $AO=OC$, $AB=DC$, $\angle 1=\angle 6$. Are triangles $AOB$ and $DOC$ congruent? If so, why?

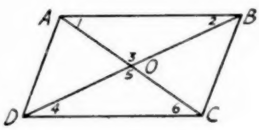



FIG. 3.

11. (Use Fig. 3.) Hyp. $\angle 1=\angle 6$, $\angle 3=\angle 5$, $AB=DC$. Are triangles $AOB$ and $DOC$ congruent? If so, why?

12. (Use Fig. 4.) Hyp. $\angle 1=\angle 2$, $\angle A=\angle C$, $BD=BE$. Are triangles $ABD$ and $CBE$ congruent? If so, why?

13. (Use Fig. 4.) Hyp. $AD=EC$, $BD=BE$, $\angle 4=\angle 5$. Are triangles $ABD$ and $CBE$ congruent? If so, why?

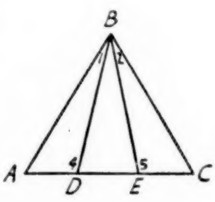



FIG. 4.

14. (Use Fig. 4.) Hyp. $\angle ABE=\angle CBD$, $AB=BC$ and $\angle A=\angle C$. Are triangles $ABE$ and $CBE$ congruent? If so, why?

TABLE 12

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 12

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 6 | 12 | 7 |
| 2 | 13 | 26 | 44 | 28 |
| 3 | 3 | 6 | 3 | 4 |
| 4 | 7 | 26 | 44 | 26 |
| 5 | 7 | 6 | 9 | 7 |
| 6 | 10 | 8 | 9 | 9 |
| 7 | 10 | 20 | 50 | 26 |
| 8 | 23 | 6 | 9 | 12 |
| 9 | 0 | 10 | 6 | 6 |
| 10 | 3 | 6 | 3 | 4 |
| 11 | 3 | 18 | 44 | 22 |
| 12 | 7 | 26 | 53 | 29 |
| 13 | 0 | 4 | 9 | 4 |
| 14 | 20 | 30 | 12 | 22 |

The high percentages in error (except in the case of Ex. 14) were on exercises in which the triangles could not be proved congruent by using only the given propositions and the given data (exs. 2, 4, 7, 11, and 12). Pupils who made errors on these exercises were not sufficiently alert to note that combinations of given sides and angles were neither *two sides and the included angle* nor *two angles and the included side*.

On the eighth exercise 12% said "No," probably due to the fact that they had already worked with many exercises in which $\angle 1=\angle 2$ and $\angle 3=\angle 4$. A change in the usual situation confused them. Note that Group A did the poorest work on this exercise.

The fourteenth exercise involved overlapping triangles. For this reason, evidently the number of errors was considerably higher than on the other exercises where the triangles were congruent under the given conditions. Judging from the number of pupils who said that triangles were congruent in exercises where they were not (under the conditions) it may well be that many guessed correctly on this exercise and the number confused by the overlapping triangles is reported too low. On 14% of the papers there was not a single "No" except on the first exercise.

We have now investigated pupils' reactions to complex figures from three points of view—in constructions, with regard to the meaning of terms, and in connection with the recognition of theorems. In all three respects we have come to the same conclusion. Changes in slight details in the figures affect the pupils' responses. Ability in connection with a simple figure does not insure ability in connection with a complex figure even though the complication may be slight. The assumption that pupils necessarily carry over what they have learned in connection with a simple figure and apply it without help to a complex figure is false. (See Chapter IX, "Transfer of Training.")

## Chapter III

## THE IF-THEN RELATIONSHIP

The if-then relationship is fundamental to postulational thinking which the pupil usually meets for the first time formally in demonstrative geometry. All propositions of geometry can be put in the if-then form. In fact, one mathematical philosopher defines mathematics as "the class of all propositions of the form, $P$ implies $Q$";[1] that is, if $P$ is true, then $Q$ is true. The concept is so fundamental that it defies explanation in terms simpler than itself. We may explain that the statement —If two sides of a triangle are equal, the angles opposite those sides are equal— means that if you construct a triangle by making two sides equal, then the angles opposite those sides will be equal without further work on your part. But even so we are using the same sentence structure, simply making the statement in more concrete form. Furthermore, if we do this we are telling the truth but not the whole truth, for the statement holds whether we construct a figure for it or not. We might say as is said in the definition above— Two equal sides in a triangle implies two equal angles opposite those sides—but then we are explaining in terms less understandable than those originally given.

The literature on the teaching of geometry is replete with statements concerning the failure of pupils to grasp the logic of geometry. It is often said that pupils begin to memorize without understanding because they do not "know what it is all about." It is not hard to understand why pupils have difficulty when many of them have no inner feeling for the meaning of the if-then relationship. When they believe, as is shown (see Chapter VII), that the two converse statements. ***If two sides of a triangle are equal the angles opposite those sides are equal,*** and ***If two angles of a triangle are equal, the sides opposite those angles are equal,*** mean little more than the bald statement, ***In an isosceles triangle two sides and two angles are equal,*** they have little basis for beginning geometry with understanding.

That many pupils do not understand the logical implication of the if-then relationship before it is developed in the geometry class, and that the growth of the concept is slow after development is begun, will be shown by the results of the tests recorded in this chapter.

### *Constructing a Figure According to Data*

The first test recorded in this chapter, Test 13, although not seemingly a direct test of pupils' understanding of the if-then relationship, nevertheless has a direct bearing upon it. Pupils were asked to construct the figure in the test making $AB$ and $ED$ perpendicular to $BD$ and $C$ the middle point of $BD$. If they bisected $BD$, that would be making $C$ the middle point of $BD$ directly. If, however, instead of bisecting $BD$, they made $AB$ equal to $ED$, that would be making $C$ the middle point of $BD$ indirectly. The second method could have been proved correct provided the pupils had had knowledge of congruence, but they did not have this knowledge. The first method puts the statement, $C$ is the middle point of $BD$, in the category of given conditions with respect to the figure; the second method puts it in the conclusion.

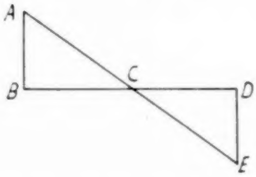
We do not claim that pupils misunderstood the difference between these two categories at this stage for they had not been discussed. We do claim that the large number of pupils who made the construction by the second method indicates very definitely that a fundamental teaching problem is involved. If a pupil makes a construction not according to the given conditions, but according to some other method that will make the figure look right in the end, he will argue that he is correct, and he can show measurements in

[1] Russell, Bertrand, *The Principles of Mathematics*, Allen and Unwin, Ltd., London, 1937, p. 1.

substantiation of his argument. The teacher knows that a fundamental principle is at stake—the difference between hypothesis and conclusion. The pupil does not know it, he does not even sense it. Therein lies the problem.

TEST 13

In the figure, $AB \perp BD$ and $ED \perp BD$. $C$ is the middle point of $BD$. Construct the figure according to these specifications.




This test was given on the twelfth day. The purpose of the test was to discover whether the pupils would construct the figure directly according to the given conditions or indirectly by some other method. Pupils had had several days' practice with construction exercises, but no preparation toward the specific purpose of this test.

TABLE 13

PERCENTAGE OF PUPILS MAKING ERRORS ON TEST 13

| Type of Error | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 33 | 44 | 41 | 40 |
| 2 | 0 | 2 | 6 | 3 |
| 3 | 0 | 8 | 12 | 7 |

The types of error indicated by the numbers 1, 2, and 3 are as follows:

1. Errors in connection with the bisector.

2. Errors in connection with the perpendiculars.

3. Other errors.

The following list shows further analysis of the errors under each of the three headings, together with the percentage of pupils in Group T making each type of error.

1(a). $BD$ not bisected, instead $AB$ made equal to $DE$, 34%.

(b). $BD$ not bisected, $AB$ not made equal to $DE$, 2%.

(c). Attempted bisection of $BD$ incorrect, 4%.

2(a). Construction of perpendiculars incorrect, 1%.

(b). Attempt to construct perpendiculars from $A$ and $E$ (unknown points), 2%.

3(a). Attempt to make $\angle A = \angle E$ in addition to the correct construction, 4%.

(b). $AE$ not through $C$, 2%.

(c). Entirely confused, 1%.

It is in the type of error numbered 1(a) that we are interested here. About one third of all the pupils thought that they were correct in making $AB = ED$ instead of making $BC = CD$. The distinction between hypothesis and conclusion is very definitely involved in this error.

*Constructing a Figure According to the Conditions of an If-Then Sentence*

On the same day (the twelfth) the following exercises (Test 14) were given as a more direct check of the pupils' understanding of the meaning of the if-then relationship.

TEST 14

1. Using rulers and compasses, construct a figure to test the truth of the following statement:

If two sides of a triangle are equal, then the angles opposite those sides are equal.

2. Using rulers and compasses, construct a figure to test the truth of the following statement:

If two angles of a triangle are equal, then the sides opposite those angles are equal.

*Caution:* Do not draw arcs on a figure *after* you have constructed it. If you wish to test the equality of any lines or angles after you have constructed a figure, do so with ruler or protractor.

Pupils knew how to construct a triangle with two sides equal. They also knew how

to construct a triangle with two angles equal. They had not been asked to do an exercise like those in Test 14 before. The purpose of the test was to discover whether, without training, pupils would sense the meaning of the if-then relationship sufficiently to construct the figures for these two converse theorems differently. The first exercise requires the pupil to construct two sides equal, and the second, two angles equal in order that the triangles be made according to the given conditions.

TABLE 14

PERCENTAGE OF PUPILS MAKING ERRORS ON TEST 14

| Percentage in Group | | | |
|---|---|---|---|
| A | B | C | T |
| 37 | 40 | 50 | 42 |

The types of error and the percentage of pupils in Group T making each type follow:

1. Made angles equal in both figures, 12%.
2. Made sides equal in both figures, 7%.
3. Constructed one figure correctly, drew the other without construction, 10%.
4. Attempt to make both the sides and the angles equal, 4%.
5. Figure omitted for one of the exercises, 9%.

Without training, 58% of the pupils constructed two sides equal in the first exercise and two angles equal on the second exercise, thus showing an understanding of the meaning of the if-then relationship sufficient to construct figures for these exercises correctly. This leaves 42%, who judging from the types of errors made, and the simplicity of the task required, showed little understanding of the relationship.

On the following day (the thirteenth) Test 14 was discussed in the following manner. The first exercise means that if you make two sides of a triangle equal, then the angles opposite those sides will turn out to be equal whether you wish them to be equal or not. You have no control over the result. When you constructed a figure to test the truth of this exercise, should you have made two sides or two angles equal? You should have made two sides equal, and then should have measured the angles with a protractor to see if the angles were equal. The second exercise was discussed in like manner.

On the fourteenth day, the day following the preceding discussion, Test 15 was given. It was of the same type as Test 14. The purpose was to discover the reactions of the pupils as a result of the explanation given. It will be noted that the figures required were more complex than those in Test 14.

TEST 15

1. Using ruler and compasses, construct a figure to test the truth of the following statement:

   If a line bisects the vertex angle of an isosceles triangle, it is perpendicular to the base.

2. Using ruler and compass, construct a figure to test the truth of the following statement:

   If a line is drawn from the vertex of an isosceles triangle perpendicular to the base, it bisects the vertex angle.

TABLE 15

PERCENTAGE OF PUPILS MAKING ERRORS ON TEST 15

| Percentage in Group | | | |
|---|---|---|---|
| A | B | C | T |
| 30 | 34 | 50 | 38 |

The types of error and the percentages of pupils making each type of error were as follows:

1. Bisected the vertex angle in both exercises, 12%.
2. Constructed perpendicular bisector of line in second exercise, 12%.
3. Made two base angles equal in both exercises, 7%.
4. Constructed perpendicular to base on both exercises, 3%.
5. Compasses not used on the second exercise, 4%.

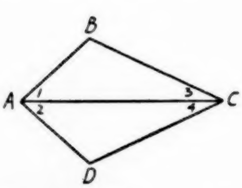
It is difficult to make a valid comparison of the results of Test 14 and Test 15. The exercises of the second are obviously more complex than those of the first, and we have shown in the preceding chapter that a pupil may be confused by a complication of figures. However, comparison of the two tests is not the important thing here. It is important to note that a large percentage of the pupils made errors even after the meaning of the if-then relationship had been explained concretely in connection with the construction of a figure. The usual procedure in geometry is to go ahead at once with the use of propositions of the if-then form on the tacit assumption that the underlying meaning is clear. Our evidence is against this assumption. The concept requires development over a period of time, and if pupils do not grasp its meaning readily in the concrete setting described here, the chances of their understanding it in the much more abstract setting of formal demonstration (without even a more careful development) certainly cannot be greater. (For a development of the meaning of the if-then relationship, see Chapter VII.)

That the growth of the concept requires careful attention over a period of time is confirmed by the results of the following test (Test 16). This test was given on the eighteenth day, six days after the subject was first introduced, and explanations similar to that after Test 14 had been given on each of these days.

TEST 16

Construct a figure like the one below to test the truth of each of the following exercises:

1. If $AB=AD$ and $\angle 1=\angle 2$, then $\angle 3=\angle 4$.




2. If $\angle 1=\angle 2$ and $\angle 3=\angle 4$, then $AB=AD$.

TABLE 16

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 16

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 16 | 21 | 14 |
| 2 | 3 | 8 | 15 | 9 |

The types of error and the percentage of pupils in Group T making each type follow:

On Ex. 1. (1) A converse construction, 12%;
(2) Faulty construction, 2%.

On Ex. 2. (1) A converse construction, 6%;
(2) Faulty construction, 3%.

We find that at least 12% of the pupils (perhaps more since the 6% wrong on the second exercise may not be among the 12% wrong on the first exercise) persisted in constructing the figure contrary to the given conditions.

### *Choosing Hypothesis and Conclusion from a Verbal Statement*

Discussion of the meaning of the if-then relationship was discontinued from this point to the thirty-eighth day. During the interval the meaning of deduction and the use of the axioms came under discussion. The procedure for explaining the meaning of the if-then relationship described in this chapter contains the germ of the method subsequently used in Part II. At this time, however, the method was not perfected and was not used consistently as it was in later classes. That there were many pupils who did not make the connection between the work already done and the procedure in drawing a figure and writing the hypothesis and conclusion in terms of the figure when a verbal statement in the if-then form is given may be seen from the table following Test 17 (given on the thirty-eighth day).

TEST 17

Draw a neat figure and write the hy-

pothesis and conclusion for each of the following exercises:

1. If a line bisects the vertex angle of an isosceles triangle, it bisects the base also.

2. If lines are drawn from any point on the perpendicular bisector of a line to the extremities of the line, they are equal.

3. If, at any point on the bisector of an angle, a line is drawn perpendicular to the bisector and extended to meet the sides of the angle, it (the perpendicular) is divided into two equal parts by the bisector.

Pupils had a working knowledge of the meaning of hypothesis and conclusion for they had for several days been proving exercises in which the hypothesis and conclusion were given explicitly in terms of a figure. As a matter of fact, they had proved the very exercises of this test when given in this way. There had been no discussion of the procedure in drawing a figure and writing the hypothesis and conclusion in terms of that figure when an exercise was given in the form of a verbal statement.

TABLE 17

Percentage of Pupils Making Errors on Each Exercise of Test 17

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 37 | 60 | 76 | 59 |
| 2 | 13 | 42 | 50 | 37 |
| 3 | 13 | 42 | 68 | 42 |

The types of error were as follows:

1. Hypothesis incomplete.
2. Hypothesis incomplete with irrelevant additions.
3. Hypothesis complete with irrelevant additions.
4. Wrong conclusion.
5. A combination of (1) and (4).
6. A combination of (2) and (4).
7. A combination of (3) and (4).
8. Wrong figure.

The distribution of errors according to type showed no general tendency in one direction. The greatest number of errors was recorded under type 6, but this type covered so many particular kinds of error (really meaning confusion) that it does not indicate any specific cause of the errors.

The results recorded in Table 17 can lead us toward only one conclusion. Fully half of the pupils did not have a feeling for the meaning of the if-then relationship sufficiently accurate to allow them to write correctly the hypothesis and conclusion in terms of a figure. It would seem that mere guesses prompted the reactions rather than any feeling for the logical implications. Pupils saw the words of the propositions and understood them singly. Then, not knowing what to do about it, they inserted the relationships indicated (and others not indicated) indiscriminately under the headings hypothesis and conclusion.

## Chapter IV

## THE MEANING OF PROOF

The proof of a proposition in geometry is deductive. Only what is implied by the data may be assumed concerning a figure. The reasoning proceeds by means of syllogistic thinking from the data to the final statement. Each intermediate conclusion becomes the basis for further deductions until the final conclusion is reached. No statement may be used as a reason unless it has been previously agreed upon. No deduction can be made correctly unless the conditions of the reason given have been completely fulfilled. A conclusion when drawn must agree explicitly with the conclusion of the authority.

All this must be assimilated consciously or unconsciously by a pupil before he can make a proof in geometry correctly—whether he writes the proof formally or thinks about it informally. Obviously a

mere explanation of these points as in the preceding paragraph, or even a lengthy expansion of the ideas, would be insufficient. They are too abstract, too far removed from the pupils' experience to make *telling* effective.

A large percentage of pupils, when they begin to study geometry have little conception of what it means to draw a conclusion from a general statement, and a specific application of it. They will draw a conclusion when the conditions are not fulfilled, and they will draw irrelevant conclusions. They do not realize that they are restricted as to data and reasons they may use, but are prone to make inferences from the appearance of a figure, or to use a reason which they have manufactured. The evidence in this chapter shows that these are basic difficulties in the learning of demonstrative geometry.

The discussion continues under the headings, Deduction, Meaning of Hypothesis, Acceptable Reasons, and Proofs of Exercises.

### *Deduction*

Deduction is seen in its simplest formal aspect in the simple syllogism which consists of three statements—a major premise, a minor premise, and a conclusion. The major premise is a general statement. The minor premise is a specific statement which fulfills the conditions of the major premise. The conclusion merely repeats the conclusion of the major premise with respect to some particular person or thing named in the minor premise.

When a pupil makes a correct deduction in geometry, he either does it by chance or memory, or he follows consciously or unconsciously the three steps of a simple syllogism. That many pupils have little realization of what it means to make a deduction is shown in the results of the following test (Test 18).

### Test 18

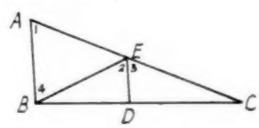
Read the directions carefully. In some exercises below, a third statement follows logically after the first two statements. In some of the exercises, a third statement does not follow logically. Whenever it is possible, write the third statement. If a third statement does not follow logically write the word "No."

I. 1. Any student who has room 323 for a home room is a senior.
2. Mary is a student and has room 323 for a home room.
3. Therefore, · · ·

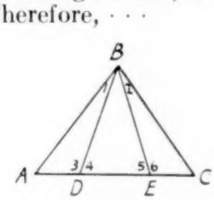
II. 1. Any student in this school who arrives after 8:30 A.M. is marked tardy.
2. Henry, a student in this school, arrived this morning at 8:32.
3. Therefore, · · ·

III. 1. All horses have four feet.
2. This animal has four feet.
3. Therefore, · · ·

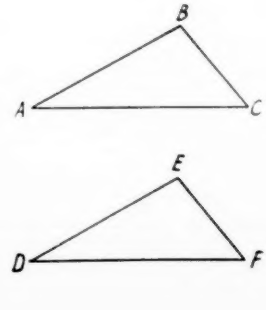
IV. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
2. Angles 1 and 4 are the angles opposite the equal sides $AE$ and $BE$ in the triangle $AEB$.
3. Therefore, · · ·




V. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
2. In triangle $DBE$, $DB=BE$.
3. Therefore, · · ·

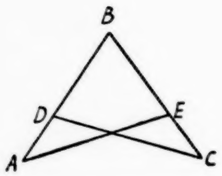



VI. 1. If two triangles have three sides of one equal to three sides of the other, they are equal.
2. $AB=DE$ and $BC=EF$.
3. Therefore, · · ·



VII. 1. Things equal to the same thing are equal to each other.
2. $a=b$ and $c=d$.
3. Therefore, · · ·

VIII. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
2. $AB=BC$.
3. Therefore, · · ·




Test 18 was given on the fifteenth day without preparation so far as the meaning of deduction is concerned. The theorem stated in Exs. IV, V, and VIII was familiar because of the work in constructions and the if-then relationship (see Test 14, page 119). The proposition in Ex. VI and the axiom in Ex. VII had not been mentioned in class. The purpose of the test was to discover whether the pupils, without training, would know when a deduction can be made from a given general statement, and when it cannot be made, and whether they could make correct deductions. The results of the test are shown in Table 18, and a detailed analysis of the errors made by Group T follows the table.

TABLE 18

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 18

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| I | 7 | 4 | 3 | 4 |
| II | 0 | 2 | 0 | 1 |
| III | 23 | 16 | 24 | 20 |
| IV | 17 | 32 | 21 | 25 |
| V | 20 | 28 | 41 | 30 |
| VI | 50 | 68 | 91 | 70 |
| VII | 30 | 58 | 71 | 54 |
| VIII | 53 | 72 | 97 | 75 |

A list of errors made by the pupils of Group T, with the percentage of pupils making each error, follows:

Exercise I.

Mary is a student, 1%.
"No," 3%.

Exercise II.

"No," 1%.

Exercise III.

This is a horse, 13%.
I do not know, 5%.
This animal has four feet, 2%.

Exercise IV.

$AE=BE$. 11%.
Triangle $AEB$ is isosceles, 5%.
"No," 5%.
The triangles are equal, 1%.
The sides opposite $\angle$s 2 and 3 are equal, 1%.
Omitted, 2%.

Exercise V.

Omitted, 6%.
Several pairs of angles equal, 15%.
$DBE$ is isosceles, 2%.
"No," 5%.
$\angle A=\angle C$, 1%.
$AB=BC$, 1%.

Exercise VI.

The triangles are equal, 32%.
$AC=DF$, 24%.
$\angle B=\angle E$, 1%.
I do not know, 5%.
The three angles are equal, 3%,
That does not make the triangles equal, the angles must be equal also, 1%.
Omitted, 4%.

Exercise VII.

$a=b$, $c=d$, 10%.
$a=c$, 12%.
$ab=cd$, 10%.
I do not know, 10%.
Omitted, 4%.
$b=d$, 2%.
$a=b$, $b=c$, $c=d$, 2%.
Things are equal to each other, 1%.
All the sides and angles are equal, 1%.
Things equal to the same thing are equal to each other, 2%.

Exercise VIII.

$\angle A = \angle C$, 58%.
Some pair of sides given equal, 5%.
I do not know, 3%.
It is an isosceles triangle, 2%.
$\angle D = \angle E$, 4%.
Omitted, 2%.
$\angle A = \angle B = \angle C$, 1%.

The first three exercises dealt with very simple everyday things. There was this difference, however, between the first two exercises and the third exercise. While a conclusion could be drawn correctly in the first two exercises, because the conditions of the general statement in each were fulfilled, no conclusion could be drawn correctly in the third exercise. This made a decided difference in the number of errors. The number of errors on the first two exercises was very small, but on the third exercise 20% of the pupils failed (13% of them saying, "Therefore it is a horse"). We are led to this conclusion, therefore—that while nearly all the pupils could make a deduction correctly in an everyday situation when there is no complication (the conditions were fulfilled), many were not sufficiently aware of what is involved to respond correctly when the conditions were not fulfilled.

The remaining exercises were geometric. In each case, the percentage of pupils who made errors was greater than the percentage in Exs. I, II, or III. The range of percentages was from 25 to 75. Exs. VI, VII, and VIII, in which the conditions of the general statement were not fulfilled, and the pupils should have written "No" meaning "No conclusion possible," caused by far the greatest number of errors.

Ex. IV was identical in form with Exs. I and II. The first premise was a familiar statement. The minor premise fulfilled the conditions of the first premise so explicitly that not even a glance at the figure was necessary in order to draw the correct conclusion. Yet 25% made errors on this exercise, as compared to 4% and 1% respectively on Exs. I and II.

Ex. V had the same first premise as Ex. IV, but in order to draw a conclusion pupils had to choose the angles opposite the given equal sides. On this exercise 30% made errors, 16% choosing the angles incorrectly.

The greatest number of errors was made on Ex. VIII, which again used the same first premise as Ex. IV. Most of the errors here, however, can be attributed to the complexity of the figure (see Chapter II). More than half of the pupils (58%) thought of the figure as a triangle with $\angle A$ and $\angle C$ opposite the equal sides and 4% said that $\angle D = \angle E$. This still leaves 13% who made other errors.

Of all the exercises, Ex. VI required the kind of reasoning most like that in the first simple exercises in congruence. In such exercises, pupils need to show that two sides and the included angle, or two angles and the included side, or three sides of one triangle, are equal to the corresponding parts of another triangle before they can draw a conclusion that the triangles are congruent. In Ex. VI we have one part of such an exercise isolated. The first premise called for the fulfillment of three conditions before a conclusion could be drawn. Only two of these conditions were fulfilled. Yet nearly three-fourths of the pupils (70%) did write a conclusion. What is more, less than half of these (32%) wrote the conclusion called for by the major premise. Many (24%) wrote as the conclusion the missing part of the conditions ($AC = DF$).

The tendency of many pupils to write as a conclusion something entirely irrelevant or something suggested by separate words or phrases of the major or minor premise is shown in the results of all the exercises except the first two. In Ex. III, 2% of the pupils made errors of this type. In Ex. IV, 18%. In Ex. V, 3%. In Ex. VI, 29%. In Ex. VIII, 7%. In Ex. VII, the showing in this respect was the worst. Although no conclusion could be drawn correctly, 54% of the pupils wrote a conclusion. The variety of conclusions

written (see list) show a rather complete confusion of ideas.

What strikes us most is the diversity of answers showing that pupils do not know when the conditions of a statement have been fulfilled nor what conclusion they should draw when the conditions have been fulfilled. They have no clear notion that they must hold themselves to the given statements and draw conclusions entirely on the basis of these statements. Anything that is suggested to them by the words or phrases or the figure gives them a lead as to what to write. They use their imaginations instead of reasoning from the given statements.

As a result of this test, we are led to the following conclusions:

(1) Although pupils can make a deduction when simple everyday situations without any complications are given, they cannot necessarily do so when there are complications even in such simple situations.

(2) Pupils need to learn that a conclusion can be drawn only when all the conditions are fulfilled.

(3) They must learn to analyze a statement to find out what the conditions are that must be fulfilled, and they must be able to see whether all the conditions have been fulfilled.

(4) They must learn that when the conditions are fulfilled the only conclusion that can be drawn is the one stated in the general statement (first premise). Pupils certainly are not ready to make demonstrations in geometry until they have mastered these things.

This test was discussed on the seventeenth day, two days after the test was given. Pupils were shown what was wrong with their papers and the right answers were given. At that time we had no general method of discussing the concepts involved. The general method was developed later (see Chapter VIII). That the explanation given was not sufficient to carry over to an unusual situation will be seen from the results of the following test (Test 19) given on the nineteenth day.

### Test 19

Write the conclusion if there is one; otherwise write "No."

I. 1. All girls have blue hair.
   2. Mary is a girl.
   3. Therefore, · · ·

II. 1. If two sides of a triangle are equal, the angles opposite those sides are right angles.
   2. In this triangle, $AB=BC$.
   3. Therefore, · · ·

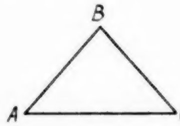



TABLE 19

Percentage of Pupils Making Errors on Each Exercise of Test 19

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 6 | 6 | 5 |
| 2 | 40 | 56 | 68 | 55 |

The types of error and the percentage of pupils making each type follow.

Ex. 1. "No." 5%.

Ex. 2. "No." 30%.

Stated that the first statement was false, and went no further, 9%.

Wrote a wrong conclusion, 14%.

Omitted, 2%.

Examples of the wrong conclusion written for the second exercise are as follows:

(1) $\angle A=\angle C$,

(2) the third angle is a right angle,

(3) $\angle A=\angle C$, but they could not be right angles,

(4) then it could not be a triangle, and

(5) it is an isosceles triangle.

In both exercises, the first premise was a false statement. On the first exercise, however, only 5% of the pupils made errors. Pupils could probably conceive of a world in which all the girls have blue hair and therefore had little trouble in drawing a correct conclusion. They could not con-

ceive of an isosceles triangle containing two right angles. What they had learned from the discussion of the preceding test was probably inhibited by the impossible situation from the concrete point of view. The result was that 55% either said that there was no conclusion, or wrote wrong conclusions as illustrated above. They had not yet learned that deduction is a sort of intellectual game in which the given statements must be followed explicitly.

The evidence so far recorded in this chapter leads the writer to believe that the cause of the logical errors so often made by pupils in proving exercises at the beginning of demonstrative geometry is fundamental. It goes back to a very hazy notion of the meaning of deduction. This statement seems only natural and might be granted without evidence except that writers of textbooks have not been sufficiently aware of this fundamental weakness to include a careful development of the meaning of deduction.

### *Meaning of Hypothesis*

In making deductions concerning a geometric figure, we are allowed to assume with respect to it only those things that are explicitly given. These given things constitute the hypothesis. From these, and these alone, may we proceed, and by means of accepted reasons and repeated deductions arrive at a conclusion.

Such restrictions are contrary to the pupil's experience before he begins demonstrative geometry. He is used to judging from appearances. It is natural for him to take the total situation—intuitive notions, appearance of symmetry, estimations of size—as a background for his arguments in geometry. To be held to the implications of only a part of a total situation is a new experience. He has to learn the meaning of hypothesis.

Again we have a fundamental concept which is not made clear by mere telling. When Test 11 (see page 114) was given, pupils had already been proving with success simple exercises involving definitions, axioms, and supplementary angles. In all of these exercises, they were of course restricted to the hypothesis with respect to a figure. In spite of this, when they were tested in a way somewhat different from their immediate experience, they showed tendencies to draw conclusions from the appearance of the figures. (See list of errors on page 115). Note especially the errors on Exs. 12 and 13.)

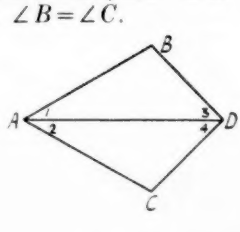
On the thirty-fourth day, just before beginning the work in congruence, another test (Test 20) was given to discover pupils' reactions to the meaning of hypothesis. As has already been said, the terms *hypothesis* and *conclusion* had been used in connection with simple exercises involving axioms and definitions, and also in exercises involving supplementary angles.

The purpose of the test was to discover whether the pupils would realize, without being specifically told in connection with this test that they must hold to the hypothesis and previously accepted propositions in choosing equal sides and angles. The only applicable authority in addition to the hypothesis was the postulate: *Any quantity is equal to itself.* This postulate had been discussed and accepted just previous to the test. It was illustrated by the use of a figure like that in the test. Pupils had been told that if they were asked if any sides of triangle $ABD$ were equal to sides of triangle $ACD$, they could say that $AD=AD$, and that the reason would be: "Any quantity is equal to itself (abbreviation, *identity*)."

Test 20

Look at the hypothesis and conclusion of the following exercises, then answer the questions.

1. Hyp. $AB=AC$, $\angle 1=\angle 2$.
   Con. $\angle 3=\angle 4$.
2. Hyp. $\angle 1=\angle 2$, $\angle 3=\angle 4$.
   Con. $\angle B=\angle C$.



Are there any lines or angles in one triangle which are equal to lines or angles in the other triangle of each exercise? If so, tell which ones they are and tell why they are equal.

TABLE 20

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 20

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 30 | 60 | 57 | 51 |
| 2 | 23 | 56 | 59 | 48 |

The distribution of errors follows:

1. Some of the correct statements omitted, no additions, correct reasons. (Ex. 1, 5%; Ex. 2, 6%.)
2. All the correct statements given, reasons correct, wrong statements in addition. (Ex. 1, 17%; Ex. 2, 17%.)
3. All the correct statements given, wrong reasons, no additions. (Ex. 1, 2%; Ex. 2, 2%.)
4. All the correct statements given, wrong reasons, wrong statements added. (Ex. 1, 4%; Ex. 2, 0%.)
5. Some of the correct statements omitted, wrong reasons, no additions. (Ex. 1, 3%; Ex. 2, 3%.)
6. Some of the correct statements omitted, correct reasons for the right statements, wrong statements added. (Ex. 1, 11%; Ex. 2, 6%.)
7. Some of the correct statements omitted, wrong reasons, wrong statements added. (Ex. 1, 10%; Ex. 2, 11%.)
8. Omitted. (Ex. 1, 0%; Ex. 2, 3%.)

Examples of errors of the above types follow:

Type 1

Ex. 1. $AB=AC$ (hyp.); $AD=AD$ (identity)

Ex. 2. $AD=AD$ (identity)

Type 2

Ex. 1. $AD=AD$ (identity); $AB=AC$, $\angle 1=\angle 2$ (hyp.); $\angle 3=\angle 4$ (parts of a bisected angle); $BD=DC$ (results of subtracting equals from equals).

Ex. 2. $AD=AD$ (identity); $\angle 1=\angle 2$, $\angle 3=\angle 4$, $\angle B=\angle C$ (hyp.).

Type 3

Ex. 1. $AD=AD$ (identity); $AB=AC$ (they are equal to the same line); $\angle 1=\angle 2$ (parts of a bisected angle).

Ex. 2. $AD=AD$ (identity); $\angle 1=\angle 2$, $\angle 3=\angle 4$ (parts of a bisected angle).

Type 4

Ex. 1. $AD=AD$ (identity); $AB=AC$, $BD=DC$, $\angle 1=\angle 2$, $\angle 3=\angle 4$. (A bisector divides an angle into two equal parts. Therefore, all lines and angles of the angle will be equal.)

Type 5

Ex. 1. $AD=AD$ (identity); $\angle 1=\angle 2$ (parts of a bisected angle).

Ex. 2. $\angle 3=\angle 4$ (parts of a bisected line).

Type 6

Ex. 1. $AD=AD$ (identity); $AB=AC$ (hyp.), $BD=CD$ (equals subtracted from equals).

Ex. 2. $\angle 1=\angle 2$, $\angle 3=\angle 4$ (hyp.); $\angle B=\angle C$ (conclusion).

Type 7

Ex. 1. $AB=AC$ (hyp.); $\angle 1=\angle 2$, $\angle 3=\angle 4$ (angles equal to the same angle). $BD=DC$ (equal to the same line).

Ex. 2. $\angle 1=\angle 2$, $\angle 3=\angle 4$ (parts of a bisected angle); $\angle B=\angle C$ (equal to the same angle).

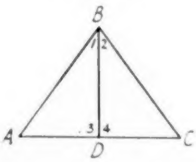
Another test, recorded here as Test 21, was given for the same purpose—to discover whether pupils had grasped the meaning of hypothesis. The test follows.

TEST 21

Cross out the parts of the following statements that are not necessarily true.

1. If $BD$ bisects $\angle B$, then I know for certain that

$$AB = BC$$
$$\angle 1 = \angle 2$$
$$\angle 3 = \angle 4$$
$$\angle A = \angle C$$
$$AD = DC$$




2. If $BD$ is perpendicular to $AC$, then I know for certain that $AB = BC$ (then followed the same statements as in the first exercise).

3. If $BD$ bisects $AC$, then I know for certain that $AB = BC$ (etc. as in Ex. 1).

(The figure was repeated for Exs. 2 and 3.)

Test 21 was given on the thirty-sixth day. Pupils had been working with exercises requiring them to hold to given conditions from the twentieth day on. In fact, just preceding this test they had proved triangles congruent in a figure like the one here, using as hypothesis the kind of data in this test. (For pupils' success in this preceding work, see Test 24 and Table 24, page 132.)

TABLE 21

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 21

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 20 | 33 | 19 |
| 2 | 3 | 12 | 35 | 17 |
| 3 | 10 | 40 | 53 | 36 |

A more detailed analysis of the errors follows:

Ex. 1. All five statements left as true, 2%.
Statements 1 and 2 left, 3%.
Statements 1 and 5 left, 6%.
Three statements left as true, 8%.

Ex. 2. All five statements left as true, 0%.
Two statements left, 10%. (6% left statements 3 and 5.)
Three statements left, 4%.
Four statements left, 3%.

Ex. 3. All five statements left as true, 4%.
Two statements left as true, 22%. (18% left statements 3 and 5.)
Three statements left as true, 6%.
Four statements left, 3%.

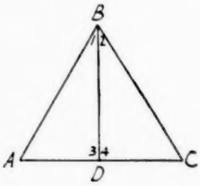
The fact that so few pupils left as true all five statements in each exercise argues for the conclusion that some attempt was made to adhere to the given facts and not to draw conclusions entirely from the appearance of the figure. In the third exercise, 18% were evidently confused between *bisector* and *perpendicular bisector*. Not so many (6%) were confused between *perpendicular* and *perpendicular bisector* in the second exercise. Even though the pupils had for some time been proving exercises deductively, many did not fully appreciate the meaning of hypothesis. When confronted with a novel situation, they did not restrict themselves to the data, but drew conclusions from the appearance of the figure.

The next test (Test 22) was given as soon as Test 21 had been completed. It was intended as a teaching device. Its primary purpose was to show those pupils who had made errors in the preceding test that the conditions of each of the first three exercises were not sufficient to determine the figure, that the figure could be drawn with various shapes and still adhere to the conditions. It was to show them objectively why they were wrong in drawing their conclusions. The test itself, however, failed in this purpose. The given figures were so potent that more pupils missed the possibility of changing them than made errors in the preceding test. The point was not made clear until the test was discussed. The test and the results are recorded here to show that the characteristics of a particular figure may have more effect upon a pupil's reasoning than the conditions given in connection with it.

TEST 22

1. If the only thing you know about this figure is that it is a triangle $ABC$, and that $BD$ must bisect $\angle B$, can you draw

the figure so that $AD$ does not equal $DC$? If so, draw it. If not, tell why not.



2. If the only thing you know about this figure is that it is a triangle $ABC$, and that $BD$ must be perpendicular to $AC$, can you draw the figure so that $AB$ is not equal to $BC$? If so, draw it. If not, tell why not.

3. If the only thing you know about this figure is that it is a triangle $ABC$, and that $BD$ must bisect $AC$, can you draw the figure so that $AB$ is not equal to $BC$? If so, draw it. If not, tell why not.

4. If the only thing you know about this figure is that it is a triangle $ABC$, and that $AB$ must equal $BC$ and $BD$ must bisect $\angle B$, can you draw the figure so that $AD$ does not equal $DC$? If so, draw it. If not, tell why not.

TABLE 22

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 22

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 13 | 32 | 39 | 29 |
| 2 | 13 | 32 | 35 | 28 |
| 3 | 47 | 52 | 68 | 55 |
| 4 | 43 | 50 | 59 | 51 |

On Ex. 1, 20% said "No," 7% drew the figure incorrectly, and 2% omitted the exercise.

On Ex. 2, 18% said "No," and 10% drew the figure incorrectly.

On Ex. 3, 38% said "No," 13% drew the figure incorrectly, and 4% omitted the exercise.

On Ex. 4, only 3% said that the figure could be redrawn. The other 48% in error were so counted because their reasons were wrong. This exercise will be discussed in the next section, entitled "Acceptable Reasons."

*Examples of reasons given on Exercise 1.* The parts of a bisected angle are equal. If you bisect "correct" and draw the lines straight, it will have to come out equal. If $BD$ bisects $\angle B$ it also bisects $AC$; no matter how large $\angle B$ is $AC$ always corresponds. One of the angles in the triangle is bisected thus making $\angle 1 = \angle 2$. Each time you enlarge or decrease $\angle 1$ and $\angle 2$ you enlarge or decrease $AD$ and $DC$. It is an isosceles triangle, and it will always be in the middle of $AC$.

*Examples of reasons given on Exercise 2.* If the perpendicular is correct and the lines straight, it will have to be equal. You can make the perpendicular any which way and $AB$ will equal $BC$. When a line is bisected it is divided into two equal parts, so $AB$ will always be equal to $BC$. Any point on $BD$ connected with $A$ and $C$ will make equal lines; the triangles are congruent.

*Examples of reasons given on Exercise 3.* The bisector makes $AB$ equal to $BC$. $AB$ and $BD$ meet at the same point to construct the triangle. $AB$ and $BC$ must make the same angle with $AD$ and $DC$ in order to meet at the same point to form a triangle, so $AB$ will equal $BC$. One line bisecting an angle will naturally be perpendicular to the other side, making the remaining sides equal.

### *Acceptable Reasons*

In proving a statement in geometry we are restricted in our reasons to the hypothesis, definitions, the unproved propositions, and the propositions we have proved up to that point. To one who has studied mathematics for some time these restrictions seem natural enough. To the beginner in demonstrative geometry they seem more or less absurd. Until the concept is built up, the beginner will give any kind of reason which seems useful to him, whether it has been previously accepted or not. "It has to be," "It looks so," "It cannot be any other way," "If those two lines go up evenly, the angles have to be equal," "Any one can see that,"—these are reasons that often seem justifiable to a beginner. He does not restrict himself to

the hypothesis and propositions which have been previously accepted, but argues from the total situation and anything in his experience, intuitive or otherwise, which comes to his mind. (See also Chapter VIII and following pages.)

Reasons given in answer to the questions of Test 22 indicate the tendency of pupils to give "made up" reasons. We have already given examples of these reasons for Exs. 1, 2, and 3. We shall now discuss the answers to Ex. 4.

Unlike the data for the first three exercises, the data for this exercise determined the figure. It could not be drawn so that $AD$ would not equal $DC$. Most of the pupils saw this. Only 3% of them said that it could be redrawn. But in spite of the fact that a large part of the class could have proved the triangles congruent correctly if they had been asked specifically to do so (see Test 24, page 132), it did not occur to 48% of them to mention the congruence of the triangles as the most acceptable reason for their answer. Twenty-five per cent of the pupils gave as a reason, "$AD$ will always 'turn out' equal to $DC$ so long as we keep to the given conditions," 23% gave manufactured reasons. They reverted to intuitions instead of using the principles of deduction, which they had been using in their formal proofs.

Examples of the manufactured reasons are as follows:

The two parts of $AC$ came out equal; it could not help it. You could not have equal sides if you drew $AD$ unequal to $DC$. If $BD$ bisects $\angle B$, then the line must go through the middle of $AC$. When an angle is formed, and then it is bisected, and both sides of the angle are equal, the base must meet both sides at $A$ and $C$. The bisection of the angle and the fact that $AB=AC$ makes $AD=DC$. It is an isosceles triangle, therefore $BD$ will always be the middle point of $AC$. It always happens to come out with $AD$ and $DC$ equal. When you connect the limits of these two lines, it will cut off an equal distance on both lines making an isoscelestriangle, and naturally when the bisector of angle $B$ meets this line, it will bisect the base of the triangle making $AD=DC$. It just cannot be done, my brain tells me that.

When pupils do not grasp the fact that their reasons in deductive proofs are restricted to those which have been accepted either by assumption or proof, they are likely to miss the underlying purpose of a deductive science. Up to this time in their experience a "proof" has been something to convince them of the truth of a statement. In elementary geometry a "proof" is not necessarily for this purpose. Many of the theorems proved could easily be accepted intuitively. (It is for this reason that some advocate the postulation of all such statements in a first course in deductive geometry.) The real purpose is to show how these theorems are dependent upon previously accepted propositions, and how they grow out of the previous statements by means of deduction and make one large logical whole. The study of demonstrative geometry has more meaning when this point is made clear by specific development.

Not so difficult for pupils to realize as the restriction of reasons just discussed is the fact that the list of reasons is increased by the proof of theorems. Nevertheless this is a point that should receive some attention. As soon as a theorem has been proved, it may be used as an authority in succeeding proofs. If this were not so, each proof would have to be reduced to the postulates and proofs would become increasingly cumbersome. Some pupils grasp this situation readily, others do not grasp it so readily, as shown by the results of the following test (Test 23).

Test 23 was given on the forty-first day immediately after the proof of the theorem, "If two sides of a triangle are equal, the angles opposite those sides are equal," had been discussed in class. The statement was made, "We may now accept this theorem and use it in proofs," but it was not emphasized, and the method of using it was not shown. The purpose of

the test was to see what percentage of the pupils would use the theorem in their proofs. Previous to this time they had become used to using four theorems, (1) When one straight line meets another so as to form adjacent angles, the angles are supplementary, (2) Supplements of the same angle or of equal angles are equal, (3) Complements of the same angle or of equal angles are equal, and (4) When two straight lines intersect, the opposite angles are equal. All other reasons had been postulates or definitions.

TEST 23

Prove the following exercise.
Hyp. $AB=BC$, $DACE$ is a straight line.
Con. $\angle 3=\angle 4$.

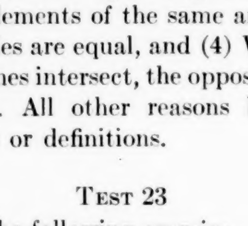



TABLE 23

PERCENTAGE OF PUPILS WHO PROVED THE EXERCISE IN TEST 23 CORRECTLY

Column 1 shows the total percentage, Column 2 shows the percentage of pupils who used the new theorem, and Column 3 the percentage of pupils who proved the exercise by means of congruent triangles.

| Group | 1 | 2 | 3 |
|---|---|---|---|
| A | 100 | 57 | 43 |
| B | 76 | 54 | 22 |
| C | 68 | 12 | 56 |
| T | 80 | 42 | 38 |

Of the 80% who proved the exercise correctly, about half (42%) used the new theorem, and about half (38%) proved it without the new theorem. Of the 20% in error, all of whom bisected angle $B$ and used the congruent triangle method, 13% said that $\angle 3$ and $\angle 4$ were corresponding parts of congruent triangles (an error under the heading of "complex figures").

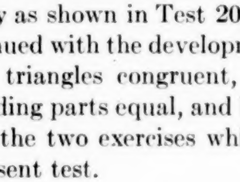
*Proofs of Exercises*

In order to show the types of errors made by pupils in formal proofs, we shall record the results of two tests. The first test, Test 24, consisted of two exercises which had been discussed in class; the second, Test 25, contained four "originals," each containing some element that was new to the pupils.

Test 24 was given on the thirty-sixth day. Pupils had had the work on the preceding day as shown in Test 20, page 127, had continued with the development proving these triangles congruent, and their corresponding parts equal, and had finally discussed the two exercises which appear in the present test.

TEST 24

1. Hyp. $BD$ bisects $\angle B$, $AB=BC$.
   Con. $\angle A=\angle C$.




2. Hyp. $AB\perp DE$, $AB$ bisects $DE$.
   Con. $\angle 1=\angle 2$.

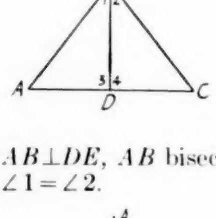



TABLE 24

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 24

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 3 | 12 | 12 | 10 |
| 2 | 20 | 28 | 41 | 30 |

In spite of the fact that these exercises had already been discussed in class, 10% of the pupils made errors on the first exercise, and 30% on the second. The errors were distributed as follows:

Errors due to "complex figure"; Ex. 1 (1%), Ex. 2 (7%).

Errors due to weakness in deduction: Ex. 1 (5%), Ex. 2 (20%).

Confused (multiplicity of errors); Ex. 1 (4%), Ex. 2 (3%).

The errors due to weakness in deduction can be further distributed as follows:

Unacceptable reasons: Ex. 1 (2%), Ex. 2 (0%).

Unfulfilled conditions; Ex. 1 (0%), Ex. 2 (13%).

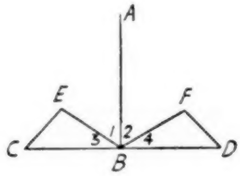
Wrong conclusion from a stated reason, Ex. 1 (3%), Ex. 2 (7%).

Although the number of pupils making errors may seem small, particularly in the first exercise, we cannot be at all sure from the results of this test that the pupils who made no errors had a full understanding of the proofs. Memory played a large role. Tests 21 and 22 (see pages 128 and 129) were given immediately after Test 24. They involved figures of the same type as those in Test 24 and the same type of hypothesis. Yet the percentages of pupils in error on these two subsequent tests were larger than those on Test 24. Teachers should beware of the false assumption that because a pupil can reproduce a proof once seen, he understands it. At the same time we see that some pupils persist in the types of error we have discussed, even when they are reproducing a proof.

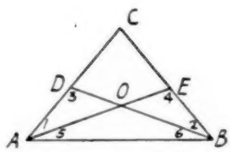
On the forty-sixth day, four days before the conclusion of the study, a final full period test was given on congruent triangles. The purpose of the test was to discover the types of errors made on original exercises. Each of the four exercises on the test required original thinking in some particular. Pupils had done an exercise in which they were required to use complementary angles in order to prove triangles congruent, but had never had an exercise where it was necessary to prove triangles congruent, and then make use of complementary angles as in Ex. 1. They had not proved triangles congruent in connection with overlapping triangles as in Ex. 2. The use of the axiom, "If equals are added to equals, the sums are equal," had never been used in connection with lines in an exercise where triangles were to be proved congruent as in Ex. 3. In Ex. 4, the necessity of proving two sets of triangles congruent was new.

TEST 25

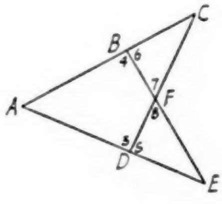
1. Prove the following exercise:
Hyp. $AB$ is the perpendicular bisector of $CD$. $CE = DF$, $\angle C = \angle D$.
Con. $\angle 1 = \angle 2$.



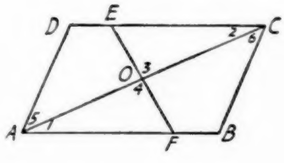

2. Prove the following exercise:
Hyp. $AC = BC$, $AD = BE$.
Con. $AE = BD$.




3. Prove the following exercise:
Hyp. $AB = AD$, $BC = DE$.
Con. $\angle C = \angle E$.




4. Prove the following exercise:
Hyp. $AB = DC$, $AD = BC$, $EF$ is a straight line passing through $O$, the middle point of the diagonal $AC$.
Con. $FO = OE$.




*Suggestion:* It is sometimes necessary to prove that more than one pair of triangles are congruent before you can arrive at your conclusion.

Pupils were allowed sixty minutes for the test. At the end of twenty minutes they were requested to begin the second exercise whether they had finished the first or not. Similarly, at the end of the next ten minutes, they were asked to begin the third exercise. Thus, all pupils tried at least three exercises.

The percentages of pupils making errors are recorded in the following table (Table 25).

TABLE 25

Percentage of Pupils Making Errors on Each Exercise of Test 25

| Exercise | Percentage in Group | | | |
|---|---|---|---|---|
| | A | B | C | T |
| 1 | 40 | 50 | 73 | 54 |
| 2 | 33 | 36 | 62 | 43 |
| 3 | 53 | 72 | 77 | 68 |
| 4 | 23 | 40 | 74 | 46 |

The errors in Group T were distributed as follows:

| | Ex. 1 | Ex. 2 | Ex. 3 | Ex. 4 |
|---|---|---|---|---|
| 1. Errors due to complexity of figure | 18% | 3% | 24% | 14% |
| 2. Conditions unfulfilled | 26% | 22% | 7% | 14% |
| 3. Unacceptable reasons | 0% | 1% | 4% | 2% |
| 4. Wrong conclusion from a stated reason | 0% | 0% | 3% | 0% |
| 5. Overdetermined construction line | 0% | 5% | 10% | 0% |
| 6. Worthless (many errors) | 5% | 1% | 0% | 0% |
| 7. Could not find a method | 5% | 11% | 19% | 14% |
| 8. No method after proving the first pair of triangles equal in Ex. 4 | — | — | — | 3% |

Note that errors due to not understanding the meaning of the if-then relationship to the extent of not being able to write the hypothesis and conclusion correctly have been avoided in this test. The hypothesis and conclusion of each exercise was given explicitly so that this error would not appear.

Throughout the study there is little, if any, evidence that the lower I.Q. groups make different kinds of errors from those made by the upper groups. In almost every case some pupils of the upper groups made mistakes which were made by the lower groups. It is true, however, that fewer pupils in the upper groups made mistakes. Only in a very few unexplainable cases did Group A make more mistakes than Group B or Group C.

By carrying this experiment over a period of fifty days, watching, recording, analyzing, and classifying pupils' reactions, we have come to the conclusion that changes in the teaching of geometry are necessary to take care of pupils' difficulties. Their difficulties are fundamental. Halfway measures will not suffice. In order that pupils may pursue the study of demonstrative geometry with understanding, and realize the aims for which it is taught, methods must be devised to help them not so much with specific difficulties as with the three fundamental difficulties we have discussed.

We see, therefore, that the statement made early in this study, that errors due to complexity of figure, and errors due to not understanding the meaning of proof, persist and account for the largest number of errors committed. The errors of omission such as those numbered 7 and 8 are beyond the scope of this study. They require an entirely different kind of analysis. The error numbered 5 is one that creeps in in certain particular kinds of exercises. It is not an error typical of all exercises.

*(To be concluded in the next issue)*

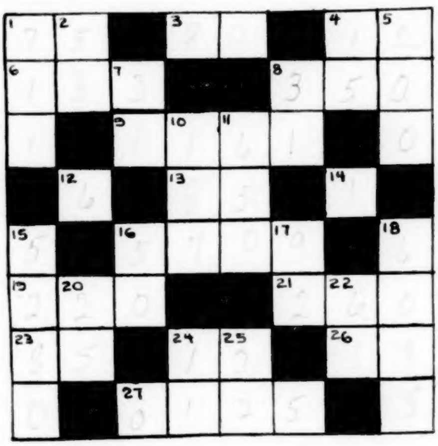
**National Council Members!**

Make your plans now to attend the summer meeting of the National Council of Teachers of Mathematics in Milwaukee, Wis. Watch for the program in a later issue of *The Teacher.*

# THE ART OF TEACHING

## Crossword Puzzle on Percentage

*By* C. R. PURDY
*Joseph Sears School, Kenilworth, Illinois*



### HORIZONTAL

1. $\frac{3}{4}$ as a per cent.
3. A salesman paid \$400 for a car and sold it at a profit of 20%. What was the profit?
4. A class of 36 pupils is $33\frac{1}{3}$% boys. How many boys are there in the class?
6. $1\frac{1}{3}$ expressed to the closest even per cent.
8. $3\frac{1}{2}$ as a per cent.
9. A newspaper report said that out of 9000 people at a football game, 12.9% were ladies. How many ladies were present?
12. Original price of a tennis racket if it sells for \$5 after a $16\frac{2}{3}$% discount is given.
13. Interest on \$1000 at 6% for 1 yr. 5 mo.
14. $\frac{1}{4}$% of 400.
16. 300% of 1903.
19. A man spent \$55 a month on food. This was 25% of his income. What was his income?
21. It is 520 miles from Chicago to St. Louis and 200 miles from Chicago to Indianapolis. What per cent of the distance to Indianapolis is the distance to St. Louis?
23. A paper boy sells 40 copies one week and increases his sales $12\frac{1}{2}$% the next week. How many papers did he sell in the two weeks?
24. $8\frac{1}{3}$% of a gross.
26. A house is insured for \$9000 for one year, at 20¢ per \$100. What is the premium per year?
27. Express $1\frac{1}{4}$% as a decimal.

### VERTICAL

1. A salesman sells \$7110 worth of goods in two months. What is his commission if the rate of commission is 10%.
2. .53 as a per cent.
4. A man's salary was increased from \$200 a month to \$230. Find the per cent of increase.
5. Bolts that cost $\frac{1}{4}$ cents each to make, are sold for 1¢ each. What is the per cent of profit?
7. Net proceeds from a sale of \$50 if the salesman is paid a commission of 38%.
8. Rate of commission paid a salesman if he is given \$93 for a \$300 sale.
10. The sale price of an article priced at \$220 if a 15% discount is given.
11. A man saves 10% of his income. What is his monthly income if he saves \$65 each month?
15. 100% of the number of feet in a mile.
16. 7 = ________% of 14.
17. 23 = 25% of ________.
18. Net price = \$7000, discount = \$915, marked price = ________?
20. A floor 20 × 22 ft. has a 9′ × 12′ rug. To the closest even per cent, what per cent of the floor is covered by the rug?
22. $30\frac{1}{2}$ is what per cent of 50?
24. Goods costing \$20 are sold for \$17.80. What is the per cent of loss?
25. A lease which was formerly \$400 per year is increased to \$488. What is the per cent of increase?

# ◆ EDITORIALS ◆

## An Appreciation of President Christofferson

It is a pleasure to give high praise to the faithful and excellent work done by the retiring president of the National Council of Teachers of Mathematics, H. C. Christofferson. While it is true that all of the officers of the Council have made notable contributions to the success of the retiring president's administration, the lion's share of the credit should go where it belongs, to the man who has not only done an unusual amount of work during the past two years, but who has also succeeded in keeping a fine *esprit de corps* among all of his colleagues. In fact, one might well refer to President Christofferson's administration as another era of good feeling. Never in the history of the Council has there been a finer spirit at an annual meeting than that evidenced at the recent St. Louis meeting. The smooth way in which the machinery has run for the last two years has been due to the untiring effort and good sense of President Christofferson. The thanks of the officers of the Council and the entire membership go to the president upon his retirement. The Council can no doubt look forward to having Professor Christofferson's advice and counsel in the days to come. The Mathematics Teacher, particularly, wishes him many more years of useful service in the cause of mathematics.

W. D. R.

## The New President of the National Council

At the recent annual meeting of the National Council of Teachers of Mathematics in St. Louis, Miss Mary A. Potter, Supervisor of Mathematics in the Public Schools of Racine, Wisconsin, was elected president of the Council for the next two years.

Miss Potter is well known to the members of the Council, since for years she has been a very active and faithful supporter of preceding administrations. She is a graduate of Lawrence College in Wisconsin and has an M.A. degree from the University of Wisconsin besides other graduate work that she has done at Harvard University and elsewhere.

After an earlier experience of five years in smaller schools, she went to Racine where she is now. She has already held the offices of state representative for the Council, director, and vice-president. The experience in these situations, her teaching experience, and her experience as an author of textbooks qualify her to fill the office of president in an excellent fashion. There is no doubt that the best interests of the Council will be served by a woman of Miss Potter's high character and qualifications. The Mathematics Teacher wishes to take this opportunity to congratulate the Council on the excellent choice they have made and to extend to Miss Potter best wishes for a most successful administration.

W. D. R.

# ◆ IN OTHER PERIODICALS ◆

*By* Nathan Lazar
*The Bronx High School of Science, New York City*

1. Albin, Joseph, "Theme-Centering the Quadratic." *High Points*, Vol. 21, No. 9, November 1939, pp. 64–67.

The writer advocates the solution of quadratics by one method only—that of factoring. By means of seven equations, worked out in great detail, he shows how equations with irrational or imaginary roots may be solved by factoring. The advantages of a "single instrument expertly handled" are pointed out.

2. "Applications of Elementary Mathematics." *The Mathematical Gazette*, 23: 338–341. October, 1939.

The following applications were sent in by readers in reply to an appeal that had appeared in the same magazine in an earlier issue.

I. Angle properties of a circle. Setting out a circular arc in railway or canal construction.

II. Properties of parallelograms, applied to various mechanisms. 1. Locomotive coupling rod. 2. Coupled switches. 3. Dual wind-shield wipers. 4. Roberval's balance. 5. Link support for front wheels of some cars and motorcycles. 6. "Lazy-tongs." 7. Wall brackets for telephones, etc. 8. Trellis fencing. 9. Lift gates. 10. Folding trays.

III. Trigonometry. Measurement of upper winds by means of pilot balloons.

Diagrams are included to illustrate most of the applications enumerated above. Hope is expressed to publish further replies in due course.

3. Hofmann, Josef Ehrenfried, "On the Discovery of the Logarithmic Series and Its Development in England up to Cotes." *National Mathematics Magazine*, 14: 37–45. October, 1939.

Logarithmic series was discovered *circa* 1667 by Newton at the age of 24, and independently by Mercator at the age of 47. Huygens and Gregory were close to the same discovery but were anticipated by the other two. This discovery was later developed by Edmund Halley, Abraham de Moivre and Roger Cotes.

The importance of logarithmic series in the development of mathematical theory is clearly pointed out. A bibliography and notes are included.

4. Hunter, Lottchen Lipp, "New Trends in the Teaching of Algebra." Bulletin of the Kansas Association of Teachers of Mathematics, Vol. 14, No. 1, October, 1939, pp. 10–14.

This article contains many suggestions to make algebra more meaningful. Relevant and helpful references are given.

5. Kleiber, Laura E., "Fundamental Considerations in the Teaching of Junior High School Mathematics." *Bulletin of the Kansas Association of Teachers of Mathematics*, Vol. 14, No. 1, October, 1939, pp. pp. 14–15.

The writer shows how mathematics can be made interesting to the student by relating it to the immediate life in or out of school. A unit on "Money, Its Use and Care" is described briefly.

6. Lenzen, V. F., "Physical Geometry." *The American Mathematical Monthly*, 46: 324–334. June-July, 1939.

"This paper may be summarized by a restatement of the relation between physical geometry and abstract geometry. Typical propositions of Euclidean geometry may be formulated as generalizations from experiences of practically rigid bodies. Such laws are expressed in terms of quantities which may be determined within limits of precision. The next step is to assume that the propositions hold exactly for a set of objects such as ideal rigid bodies. Propositions with a precisely defined content may be reduced to a set of axioms from which theorems can be deduced. . . . Once we have the concept of abstract geometry, it is possible to create new abstract geometries and then seek physical interpretations of them. . ."

7. Loria, Gino, "Triangles Équilatéraux Dérivés d'un Triangle Quelconque." *The Mathematical Gazette*, 23: 364–372, October, 1939.

A famous mathematician states and proves

the following two theorems which should be of interest to teachers of geometry:

(a) The centers of the equilateral triangles constructed *externally* on the sides of any triangle are the vertices of another equilateral triangle.

(b) The centers of the equilateral triangles constructed *internally* on the sides of any triangle are the vertices of another equilateral triangle.

In the second part of the article, the author discusses some questions connected with the Morley triangle. A short bibliography is included.

8. Read, Cecil B., "Standardized Test for a Final Examination in College Algebra." *Bulletin of the Kansas Association of Teachers of Mathematics*, Vol. 14, No. 1, October, 1939, pp. 9–10.

Some comments, observations, and speculations on the results obtained by giving a single uniform examination—the Cooperative Examination in College Algebra—in place of the customary, departmental final examination.

9. Schorling, Raleigh, "The Place of Mathematics in General Education." *School Science and Mathematics*, 40: 14–26. January, 1940.

The writer believes that "the needs of the intelligent citizen suggest that the curriculum should be designed to give the pupil (1) effective skills in computation; (2) the control of about fifty basic mathematical concepts that contribute to the understanding of the general reader of the public press; (3) such information and attitudes as will give a maximum security to a family with a given income; and (4) improvement in reflective thinking."

Each of these aims is discussed in some detail.

10. Shreve, D. R., and Keller, M. W., "A Note on the Graphing of the Equation Homogeneous in $X$ and $Y$." *School Science and Mathematics*, 40: 48–50. January, 1940.

The purpose of this note is to recall to teachers of analytic geometry some of the simpler geometric interpretations of the algebraic theory applicable to equations of a higher degree homogeneous in the two variables.

11. Travers, J., "Rules for Bordered Magic Squares." *The Mathematical Gazette*, 23: 349–351. October, 1939.

The author presents rules for the construction of odd-bordered squares and even-bordered squares. These rules, he believes, are published for the first time.

12. Yates, Robert C., "To Have and to Hold." *National Mathematics Magazine*," 14:2–4. October, 1939.

A plea to teachers of mathematics to make their subject vital and absorbing. Interesting projects, devices, and constructions are mentioned.

13. Whitmer, Edith F., "Youth Speaks for Mathematics." *School Science and Mathematics*, 40: 41–46. January, 1940.

One hundred and five boys and girls studying mathematics in the Community High School, Winchester, Illinois, co-operated in attempting to determine what immediate uses they were finding for mathematics and to what extent it was being correlated with other subjects they were studying.

The results of this investigation are tabulated in fourteen categories. Some of the replies are quoted verbatim in this report.

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## Spring Thoughts

*By* Anna L. Fellroth
*Minnehaha Academy, Minneapolis, Minn.*

Little robin, you've made your nest so firmly round.
Without a compass I wonder how the circle you have found?
With care you chose the fork of branches to place your nest.
How did you know triangular rigidity secures it best?
Where did you learn the postulate of two points determining a line
Which enables you to return each spring to this tree of mine?
Ah! I hear your song of praise, little singing mathematician,
Which you in thanks give forth to your Teacher, the Lord of creation.

# NEWS NOTES

Dr. Guy Stevenson, Head of the Mathematics Department and Dean of the Graduate School of the University of Louisville, addressed forty-three mathematics teachers who met with the Kentucky Education Association at Louisville, November 18. He discussed the problems facing secondary school teachers of mathematics particularly those in Kentucky.

The following resolution was adopted by the mathematics group and sent to the Dean and President of each State Teachers College in Kentucky, to the Dean of the College of Liberal Arts and the President of Louisville University and the University of Kentucky, and to the State Department of Education.

Resolved: We believe that mathematics should occupy a fundamental place in any scheme for the general education of the youth of our state. Mathematics is an essential part of the foundation for the technical training required for a large number of occupations from which the grade pupil and the high school pupil will likely choose their life's work. We believe that the door of opportunity to these occupations should be kept open to every child in the state. This can only be done by making adequate provision for him to acquire the necessary mathematical foundation for such technical training as a part of his general education in the grades and in the high school.

We believe, further, that the study of mathematics affords a type of cultural and intellectual training given by no other subject. In no other subject is there such opportunity to test the accuracy of conclusions and to detect errors and imperfections in processes and results of thinking as is found in the study of mathematics. The student of this subject thus acquires an appreciation for exact reasoning and comes in contact with the best example of a logical system to be found anywhere in his school course.

In the study of mathematics the student has the only opportunity provided in his school course to learn to solve the increasing number of actual quantitative problems that affect the personal lives of our citizens in an increasingly complicated social and industrial system.

Mathematics has played a fundamental role in the development of modern industrial civilization, and we believe that every educated person should have some knowledge and appreciation of a subject that has made such vital contributions to human progress and which must, obviously, be an essential factor in any future advancement.

In consideration of these and many other recognized values of mathematics as a school subject, we most respectfully and most earnestly request the curriculum-making agencies of the state to provide for every pupil in the grades and in the secondary schools of the state full opportunity to take advantage of these values. In order that this may be more fully accomplished, we urge that students in our teacher training institutions who are preparating to teach mathematics either in the elementary grades or in the high school be given adequate preparation for effective teaching of the subject. We urge that they be given sufficient training in mathematics for them to acquire an appreciation of its true value and usefulness as a school subject. We urge, further, that all students in our public institutions of higher learning be given full opportunity to avail themselves of the contributions that the study of mathematics makes to culture and to a broad, liberal education.

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Teachers College, Columbia University, will offer the following courses in the teaching of mathematics this summer, July 8 through August 16:

By Professor C. B. Upton: Teaching Arithmetic in Primary Grades, first three grades, July 8–26; Curriculum Problems in Elementary Arithmetic, July 8–26. Professor E. R. Breslich: Teaching and Supervision of Mathematics, junior high school; Teaching and Supervision of Mathematics, senior high school. Professor John R. Clark: Teaching Geometry in Secondary Schools; Demonstration Class in Plane Geometry. Professor Carl N. Shuster: Modern Business Arithmetic; Field Work in Mathematics. Miss Ethel Sutherland: Teaching Arithmetic in Intermediate Grades, fourth, fifth and sixth grades, July 8–26. Dr. John Swenson: Professionalized Subject Matter in Senior High School Mathematics; Demonstration Class in High School Calculus. Dr. Nathan Lazar: Teaching Algebra in Junior High School; Logic for Teachers of Mathematics. Dr. R. R. Smith: Teaching Algebra in Secondary Schools; Teaching of Trigonometry, Solid Geometry, and Advanced Algebra.

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The Wichita, Kansas, Mathematics Association held a dinner meeting in November at which Professor H. C. Christofferson, President of the National Council of Teachers of

Mathematics, was the guest speaker. "Frontiers in Mathematics Education" was the subject of his talk which included a discussion of functional relationships and generalization.

Dr. Christofferson gave addresses in the following cities: in St. Louis, to the State Teachers Association, November 18; Wichita, November 18; Oklahoma City, November 20; and Tulsa, November 21.

At the first meeting of the current school year, the Wichita Mathematics Association heard the report of the San Francisco meeting by Miss Lorena Cassidy, the official delegate. Miss Isis Woodward also gave a talk on "Social Life and the Schools in Puerto Rico."

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The Women's Mathematics Club of Chicago and Vicinity held its first meeting of the current year on October 7. Miss Marie Plapp, president, gave the first of a series of five minute talks to be given by members. Her topic was "Mathematics in New York World's Fair." Luncheon was followed by a tour of the Rosenwald Museum of Science and Industry. A member of the staff gave a brief explanation of the exhibits in the Museum that are of particular interest to mathematics teachers.

The second meeting of the Woman's Mathematics Club of Chicago and Vicinity was held in Chicago's "China Town" at Won Kow Restaurant, November 4. A five minute talk was given by Miss Edith Levin of Englewood High School, speaking on "Uses of Mathematical devices in Teaching." Doctor Philip Fox, director of the Rosenwald Museum of Science and Industry, was the guest speaker taking for his subject "Mathematics in Science and Industry."

DOROTHY B. LANDERS

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The very active Men's Mathematics Club of Chicago and the Metropolitan Area has had excellent programs for its recent monthly meetings.

Dr. F. W. Godwin, Director of the Chemical Engineering Research Division of the Research Foundation of the Armour Institute of Technology, spoke at the December dinner meeting on "The Antarctic Snowcruiser." His lecture was illustrated with slides.

The January meeting had for speakers U. C. Foster on "Projects in Solid Geometry." Dean Ovid W. Eshbach, of Northwestern University, spoke on "The New Technological Institute of Northwestern University and Mathematics Preparation for Engineering." Dean Eshbach has been with the American Telegraph and Telephone Company, Massachusetts Institute of Technology, and is an editor of an engineering handbook.

The February meeting was held jointly with the Women's Mathematics Club of Chicago. E. C. Moline, Switching Theory Engineer of the Bell Telephone Laboratories, New York, spoke on "Mathematics in the Telephone Industry." "A Hurricane's Challenge" a two reel feature of the 1938 New England Hurricane was included in the program.

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*American Journal of Physics* is the new title of the bi-monthly publication known since its inception in 1933 as *The American Physics Teacher*, according to an action taken recently by the American Association of Physics Teachers concerning its official journal. Remaining under the editorship of Professor Duane Roller, of Hunter College, and under the publication management of the American Institute of Physics, the journal will continue to stress the educational, historical, socio-economic and philosophic aspects of physics, and the instruction of students who take physics as part of a liberal education as well as those who specialize in the science.

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The Organizing Committee announces with regret that the International Congress of Mathematicians which was scheduled to be held in Cambridge, Massachusetts, in September 1940, is postponed until a more favorable time. Due notice will be given of any arrangements to hold the Congress at a later date.

R. G. D. RICHARDSON, *Secretary*

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The Texas State Teachers Association Mathematics Section has elected its chairman for 1940–41, Professor S. M. Sewell, of the Southwest Texas State Teachers College. Miss Genelle Bell, Beeville, Texas, has been elected secretary-treasurer. The State Association meets each year in November in one of the large cities of Texas. The 1939 meeting was in San Antonio.

ELIZABETH DICE

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The Mathematical Association of America announces the Third Annual William Lowell Putnam Mathematical Competition. A competitive examination in collegiate mathematics was given Saturday, March 2; and according to the results, medals, cash prizes, and a thousand dollar scholarship are to be awarded. Harvard University will award this annual $1000 William Lowell Putnam Prize Scholarship to one of the first five contestants in the competition.

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The American Association of Junior Colleges has received a grant of $25,000 from the General

Education Board, of New York City, to finance a series of exploratory studies in the general field of terminal education in the junior college. Approximately 500 accredited junior colleges are now found in the United States besides another hundred which are not yet thus recognized.

About two-thirds of the 175,000 students enrolled in these institutions do not continue their formal education after leaving the junior college. The new study will be concerned particularly with courses and curricula of a semi-professional and cultural character designed to give this increasing body of young people greater economic competence and civic responsibility. There is increasing evidence that existing four-year colleges and universities are not organized adequately to meet the needs of a large part of this significant group.

It is anticipated that the exploratory study will reveal the need and the opportunity for a series of additional studies and experimental investigations and demonstrations which may cover several years of continuous effort.

Immediate responsibility for the study will be vested in an executive committee consisting of Rosco C. Ingalls, Chairman, Doak S. Campbell, and Byron S. Hollinshead. The Director of the study will be Walter Crosby Eells, Executive Secretary of the American Association of Junior Colleges, Washington, D. C.

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The first two numbers of the new international journal *Mathematical Reviews* have been sent to 15,000 libraries, universities, scientific societies, and scholars in all parts of the world.

Sponsored by the American Mathematical Society and the Mathematical Association of America, with the aid of the Carnegie Corporation and the Rockefeller Foundation, the new *Mathematical Reviews* will seek to attain worldwide acceptance among mathematicians on the globe's six continents.

The first issue contained 196 reviews of papers written by mathematicians in all parts of the world, concerning research on such topics as abstract algebra, the theory of groups, the theory of functions of complex variables, celestial mechanics, relativity, and numerical and graphical methods.

Editors of the new journal have subscribed to about 400 other periodicals dealing with the field of mathematics, in order to keep in touch with the vast number of mathematical papers being issued everywhere. No Polish or Czech publications have come to the editoral offices at Brown University, and the war has affected European scholarship, in general, but it is expected that 600 periodicals will be received and scanned in normal times.

The editors have 370 collaborators in all parts of the world. Papers chosen from the periodicals for review in the new journal are forwarded to selected collaborators for summary. All papers reviewed in the new journal are also copied on film or photographed, and are available at cost to all scholars interested in receiving the complete text from which the review was made.

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The annual meeting of the Northern California Section of the Mathematical Association of America was held on January 27 at the University of California, Berkeley.

Some of the important addresses were: "A Simple Mathematical Theory of Economic Relief," by Professor G. C. Evans, of the University of California. "Mathematics and the Constructive Arts" by Professor W. F. Durand, of Stanford University. "Geometric Representation of Certain Magnetic Fields," by Professor F. R. Morris, Fresno State College. "Shall We Defer the Teaching of Algebra to the Tenth Grade?" by Miss Harriet Welch, Lowell High School, San Francisco. "Some Difficulties with Mathematics in a Core Curriculum," by Dr. Vern James, Menlo Junior College. "A General Solution of $x_1^2+x_2^2+\cdots x_n^2=m^2$," by A. L. McCarty, San Francisco Junior College.

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"Arithmetic—An Art and a Science" was the theme of the morning meetings of the Mathematics Section of the Illinois High School Conference held November 3. The following addresses were given:

"Arithmetic in the Changing Elementary Schools of Today," by R. O. Gibbons, Principal of Franklin School, Quincy. "The Modern Mathematical Background and Training of the Elementary School Teacher," by W. B. Storm, head of the department of mathematics, Northern Illinois State Teachers College, DeKalb. "The Dilemma of the High School Teacher" by Edwin W. Schreiber, Western Illinois State Teachers College, Macomb.

The afternoon sessions included the showing of a play "Mathematics for the Millions" by Milton Kaletsky, given by the students of the University High School under the direction of Dr. Henrietta Terry.

Other features were addresses: "Leadership by High School Mathematics Teachers," by Miss Ruth Lane, University High School, Iowa City; and "Teaching Mathematics Today for the World of Tomorrow," by Dr. D. W. Kerst, Physics Department, University of Illinois.